Continuous Mathematics (Sheet #3)

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1. (a) $a_0 = \frac{1}{2}\int\limits_{-2}^{2} g(x)dx = \frac{1}{2}\ x|_{-1}^{1} = 1$

   $a_n = \frac{1}{2}\int\limits_{-2}^{2} g(x)\cos\frac{n\pi x}{2}dx = \sin\frac{n\pi x}{2}\Big|_0^1 = \sin\frac{n\pi}{2}$

   $b_n = 0$ because g is even.

   $f(x) = \frac{1}{2} + \sum\limits_{n=1}^{\infty}\left(\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2}\right).$

   (b) $f(-1) = \frac{1}{2} + \sum\limits_{n=1}^{\infty}(\sin\frac{n\pi}{2}\cos\frac{n\pi}{2}) = \frac{1}{2}$

2. $b_n = \frac{1}{a}\int_{-a}^{a} f^o(x)\sin\frac{n\pi x}{a}dx = \frac{1}{2\pi}\int\limits_{-2\pi}^{2\pi} f^o(x)\sin\frac{nx}{2}dx = \frac{1}{2\pi}(-2\pi\int\limits_{-2\pi}^{-\pi}\sin\frac{nx}{2}dx$
$-\int\limits_{-2\pi}^{-\pi} x\sin\frac{nx}{2}dx + \int\limits_{-\pi}^{\pi} x\sin\frac{nx}{2}dx + 2\pi\int\limits_{\pi}^{2\pi}\sin\frac{nx}{2}dx - \int\limits_{\pi}^{2\pi} x\sin\frac{nx}{2}dx) =$
$\frac{1}{2\pi}\left(\int\limits_{-\pi}^{\pi} x\sin\frac{nx}{2}dx + 4\pi\int\limits_{\pi}^{2\pi}\sin\frac{nx}{2}dx - 2\int\limits_{\pi}^{2\pi} x\sin\frac{nx}{2}dx\right) =$
$\frac{1}{2\pi}\left(\left(\frac{4}{n^2}\sin\frac{nx}{2} - \frac{2x}{n}\cos\frac{nx}{2}\right)\Big|_{-\pi}^{\pi} - 2\left(\frac{4}{n^2}\sin\frac{nx}{2} - \frac{2x}{n}\cos\frac{nx}{2}\right)\Big|_{\pi}^{2\pi} + 4\pi\cos\frac{nx}{2}\Big|_{2\pi}^{\pi}\right) =$
$\frac{1}{2\pi}\left(\frac{4}{n^2}(\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} - 2\sin(n\pi) + 2\sin\frac{n\pi}{2})\right) - \frac{2}{n}(\pi\cos\frac{n\pi}{2} + \pi\cos\frac{n\pi}{2} + 4\pi\cos(n\pi) - 2\pi\cos\frac{n\pi}{2} -$
$2\pi n\cos\frac{n\pi}{2} + 2\pi n\cos(n\pi)) =$
$\frac{1}{2\pi}(\frac{\pi}{n}\cos\frac{n\pi}{2}) - \frac{8}{n^2}\sin n\pi + \frac{16}{n^2}\sin\frac{n\pi}{2} = \frac{8}{n^2\pi}\sin\frac{n\pi}{2} = \frac{8}{(2k+1)\pi}(-1)^k.$

3. (a) $x = \int te^t dt = te^t - \int e^t dt = (t-1)e^t + C$

   (b) $y = \int\frac{dx}{x(x-1)(x+1)} = \int\frac{A}{x}dx + \int\frac{B}{x-1}dx + \int\frac{C}{x+1}dx$
   $A(x^2-1) + B(x^2+x) + C(x^2-x) = 1 \implies A = -1, B = 0.5, C = 0.5.$
   So $y = -\ln(x) + 0.5\ln(x-1) + 0.5\ln(x+1) + C$.

   (c) $\int ydy = \int x^2dx \implies \frac{1}{2}y^2 = \frac{1}{3}x^3 + A \implies y = \sqrt{\frac{2}{3}x^2 + 2A}.$

4. $\frac{dy}{dx} = 2v^2 + v.$
$\frac{dy}{dx} = v + x\frac{dv}{dx} \implies \frac{dv}{dx} = \frac{2v^2}{x} \implies \int\frac{1}{x}dx = \int\frac{1}{2v^2}dv \implies \ln|x| = -\frac{1}{2v} - C \implies -2(\ln|x| - C) = \frac{x}{y} \implies y = \frac{x}{-2(\ln|x|-C)}$

5. (a) $\frac{d}{dx}(ye^{\ln x}) = \frac{d}{dx}(yx) = 0 \implies yx = A.$
   We know $y = 2$ for $x = 2$, so $A = 4$ and $y = \frac{4}{x}$.

   (b) $\frac{dy}{dx} + \frac{2}{x^3}y = e^{x^{-2}-x^3} \implies \frac{d}{dx}(ye^{-x^{-2}}) = e^{x^{-2}-x^3}.$
   $ye^{-x^{-2}} = \int e^{x^{-2}-x^3}$

6. (a) $y = e^{\alpha x}$

   $\alpha^2 + 1 = 0 \implies y = Ae^{ix} + Be^{-ix} = (A+B)\cos x + (A-B)\sin x$. From the boundary conditions we get $A + B = 0$ and $A - B = 1$ therefore $y = \sin x$.

   (b) $\alpha^2 - 6\alpha + 9 = 0 \implies y = e^{3x}(A + Bx)$. From the boundary conditions we get $A = 1$ and $-1 = e^3(1 + B) \implies B = -e^{-3} - 1$.

7. (a) Homogeneous equation gives us $\alpha = 0$ so $y = A + Bx$. From the boundary conditions we get $A = 1, B = -2$. Particular solution is $y = e^{\pi x}$. General solution is $y = e^{\pi x} + 1 - 2x$.

   (b) $\alpha^2 + \pi^2 = 0$ so $\alpha = \pm i\pi$. We get $y = A\cos x\pi + B\sin x\pi$. From the boundary conditions, $A = 1$ and $-A = -1$, so $A = 1$ and B can take any value.

   (c) Same as above with different boundary conditions. We get $A = 1$ and $-AA = 0$, a contradiction.

9.

10. (a) $x = e^t$.

    $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}e^t = \frac{dy}{dx}x$.

    $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0$

    $\alpha = 1$

    We get $y = e^x(A + Bx)$.

    (b)