# Continuous Mathematics (Sheet #4)

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1. (a) Let $w(x,t) = v(x,t) - u(x,t) = -(U_0 + (U_1 - U_0)x)$. We have $\frac{\partial w}{\partial t} = 0$ and $\frac{\partial^2 w}{\partial x^2} = 0$. Hence $v(x,t)$ satisfies the given equation on the given conditions.

   The boundary conditions become $v(0,t) = u(0,t) - U_0 = 0$ and $v(1,t) = u(1,t) - U_1 = 0$.

   The initial conditions become $v(x,0) = u(x,0) - (U_0 + (U_1 - U_0)x) = -(U_0 + (U_1 - U_0)x), 0 < x < 1$.

   (b) If $v(x,t) = X(x)T(t)$ we have $\frac{\partial v}{\partial t} = X(x)\frac{dT}{dt}$, $\frac{\partial v}{\partial x} = T(t)\frac{dX}{dx}$ and $\frac{\partial^2 v}{\partial x^2} = T(t)\frac{d^2X}{dx^2}$.

   Therefore $X(x)\frac{dT}{dt} = DT(t)\frac{d^2X}{dx^2} \iff \frac{dT}{dt}\frac{1}{DT(t)} = \frac{d^2X}{dx^2}\frac{1}{X(x)} = \lambda$.

   Analogously to the lecture notes we find that $\lambda \geq 0$ only gives us the trivial solution. If $\lambda < 0$ then we can write $\lambda = -k^2$. Then, $\frac{d^2X}{dx^2} + k^2X(x) = 0$ so $\alpha^2 + k^2 = 0$ so $\alpha = \pm ik$. The solutions are $X(x) = A\cos(kx) + B\sin(kx)$.

   From the boundary conditions we have $v(0,t) = 0 \implies X(0)T(t) = 0 \implies X(0) = 0$ so $A\cos(0) + B\sin(0) = A = 0$ and similarly $X(1) = 0$ so $B\sin(k) = 0$ therefore either $B = 0$ (which gives us the trivial solution) or $\sin(k) = 0$. Thus, $k = n\pi$ and $\lambda = -n^2\pi^2$.

   Now we have $\frac{dT}{dt}\frac{1}{DT(t)} = -n^2\pi^2 \implies \frac{T'(t)}{T(t)} = -n^2\pi^2 D = \ln'(T(t)) = -n^2\pi^2 D = \ln(T(t)) = -n^2\pi^2 Dt + C$ so $T(t) = Ee^{-n^2\pi^2 Dt}$.

   Thus the solution for some fixed $n$ is $B\sin(n\pi x)Ee^{-n^2\pi^2 Dt}$ and the general solution is $v(x,t) = \sum\limits_{n=1}^{\infty} A_n \sin(n\pi x)e^{-n^2\pi^2 Dt}$ where $A_n = B_nE_n$.

   Now $u(x,t) = v(x,t) + (U_0 + (U_1 - U_0)x) = (U_0 + (U_1 - U_0)x) + \sum\limits_{n=1}^{\infty} A_n \sin(n\pi x)e^{-n^2\pi^2 Dt}$.

   We can find $A_n$ from the initial conditions. $u(x,0) = 0$ therefore $0 = (U_0 + (U_1 - U_0)x) + \sum\limits_{n=1}^{\infty} A_n \sin(n\pi x)e^{-n^2\pi^2 Dt}$ so $\sum\limits_{n=1}^{\infty} A_n \sin(n\pi x)e^{-n^2\pi^2 Dt}\sin(m\pi x) = -(U_0 + (U_1 - U_0)x)\sin(m\pi x)$.

   We can integrate and get $\sum\limits_{n=1}^{\infty} A_n e^{-n^2\pi^2 Dt}\int\limits_0^1 \sin(n\pi x)\sin(m\pi x)dx =$

   $-\int\limits_0^1 (U_0 + (U_1 - U_0)x)\sin(m\pi x)dx$.

   If $n \neq m$ then $\int\limits_0^1 \sin(n\pi x)\sin(m\pi x)dx = \int\limits_0^1 [\cos((n-m)\pi x) - \cos((n+m)\pi x)]dx = \sin((n-m)\pi x) - \sin((n+m)\pi x)|_0^1 = 0$.

   If $n = m$ then $\sum\limits_{n=1}^{\infty} A_n e^{-n^2\pi^2 Dt}\int\limits_0^1 \sin^2(n\pi x)dx = -\int\limits_0^1 (U_0 + (U_1 - U_0)x)\sin(n\pi x)dx$.

   $\int\limits_0^1 \sin^2(n\pi x)dx = \frac{1}{2}\int\limits_0^1 1 - \cos(2n\pi x)dx = \frac{1}{2}(x - \frac{\sin(2n\pi x)}{2n\pi}\Big|_0^1) = \frac{1}{2}$.

   So $\frac{1}{2}\sum\limits_{n=1}^{\infty} A_n e^{-n^2\pi^2 Dt} = -\int\limits_0^1 (U_0 + (U_1 - U_0)x)\sin(n\pi x)dx$.

   (Now what?)

2. Suppose there are two solutions $T_1, T_2$ and take $\Delta = T_1 - T_2$. Then $\Delta$ would be a solution to the homogeneous equation.

So $\frac{\partial \Delta}{\partial t} = D\frac{\partial^2 \Delta}{\partial x^2}$ with boundary conditions $\Delta = 0$ for $x = 0, t > 0$ and $t = 0, 0 < x < L$.

We define $E(t)$ by $\int_0^L [\Delta(x,t)]^2 dx$ so $E(0) = 0, E(t) \leq 0$.

We get $\frac{dE}{dt} = \frac{d}{dt}\int_0^L [\Delta(x,t)]^2 dx = \int_0^L \frac{\partial}{\partial t}[\Delta(x,t)]^2 dx = \int_0^L 2\Delta \frac{\partial \Delta}{\partial t} dx = \int_0^L 2\Delta(D\frac{\partial^2 \Delta}{\partial x^2} + f(x,t))dx = -\int_0^L 2D\left(\frac{\partial \Delta}{\partial x}\right)^2 dx + 2\int_0^L \Delta f(x,t)dx \leq 0$. (Why?)

3. (a) We get $\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 1 + r^2$.
   (b) The equation is $\frac{d}{dr}(r\frac{du}{dr}) = r + r^3 \implies r\frac{du}{dr} = \frac{1}{2}r^2 + \frac{1}{4}r^4 + C \implies u = \frac{1}{4}r^2 + \frac{1}{16}r^4 + C\ln|r| + D$.

4. (a) $\frac{d^2y}{dx^2} = x \implies \frac{dy}{dx} = \frac{1}{2}x^2 + C \implies y = \frac{1}{6}x^3 + Cx + D$
   We know $y = 0$ at $x = 0, 1$ therefore $0 = D$ and $0 = \frac{1}{6} + C$. So $y = \frac{1}{6}x^3 - \frac{1}{6}x = \frac{x^3 - x}{6}$.
   (b) We know that the second derivative of a function is the average of the forward and backward Euler method. Therefore:
   $y''(x_n) = \frac{y_{n-1} - 2y_n + y_{n+1}}{h^2}$.
   We have $x_n = nh$ therefore $\frac{y_{n-1} - 2y_n + y_{n+1}}{h^2} = nh \iff y_{n-1} - 2y_n + y_{n+1} = nh^3 \iff y_n - 2y_{n+1} + y_{n+2} = (n+1)h^3$.
   (c) Auxiliary equation: $\alpha^2 - 2\alpha + 1 = 0 \implies \alpha = 1$ so $y = C + nD$.
   (d) The homogeneous solution uses $n^0$ and $n^1$ so our first guess would be $Cn^2$.
   We get $C(n+2)^2 - 2C(n+1)^2 + Cn^2 = h^3(n+1) \implies 2C = h^3(n+1)$ so $C = \frac{n+1}{2}h^3$, not a constant.
   Our next guess is $Cn^3 + Dn^2$, and we can check that $C = \frac{h^3}{6}, D = 0$ give us $\frac{h^3}{6}((n+2)^3 - 2(n+1)^3 + n^3) = \frac{h^3}{6}(n^3 + 6n^2 + 12n + 8 - 2n^3 - 6n^2 - 6n - 2 + n^3) = \frac{h^3}{6}(6n + 6) = h^3(n+1)$
   (e) The general solution is $\frac{h^3}{6}n^3 + nD + C$.
   Using the boundary conditions we get $C = 0$ and $\frac{h^3}{6}N^3 + ND = 0 \iff \frac{1}{6} + ND = 0 \iff D = -\frac{1}{6N}$
   Finally we have $\frac{h^3}{6}n^3 - \frac{1}{6N}n = \frac{x_n^3 - x_n}{6}$.

5. Standing waves, from the lecture notes: $u(x,t) = W(x+ct) + W(x-ct)$ where $W(\theta) = \frac{1}{2}\sum_n A_n \sin(\frac{n\pi}{L}\theta)$ and $A_n = 0$ for $n$ even; $A_n = \left(\frac{8d}{\pi^2 n^2}\right)(-1)^{\frac{n-1}{2}}$ for $n$ odd.

We also have $d = 6$, so $u(x,t) = \frac{1}{2}\left(\sum_n \frac{48}{\pi^2 n^2}(-1)^{\frac{n-1}{2}}\left(\sin(\frac{nx\pi}{100} + \frac{nt\pi}{10}) + \sin(\frac{nx\pi}{100} - \frac{nt\pi}{10})\right)\right)$ where $n$ is odd. $u(x, 2.5) = \frac{1}{2}\left(\sum_n \frac{48}{\pi^2 n^2}(-1)^{\frac{n-1}{2}}\left(\sin(\frac{nx\pi}{100} + \frac{n\pi}{4}) + \sin(\frac{nx\pi}{100} - \frac{n\pi}{4})\right)\right)$ etc.

6. $v_n = v_{n-1} - \Delta t g \implies v_n = -\Delta t g n$.

$z_n = z_{n-1} + \Delta t\frac{1}{2}(v_{n-1} + v_n) = z_{n-1} + \Delta t\frac{1}{2}(-\Delta t g(2n-1)) = z_{n-1} - \Delta t^2\frac{1}{2}g(2n-1) = -\Delta t^2\frac{1}{2}g\sum_{k=1}^{n}(2k-1) = -\Delta t^2\frac{1}{2}gn^2$.

Checking against the real equations using $t = \Delta t n$, we get $v = -gt = -\Delta t g n = v_n$ and $z = -\frac{1}{2}gt^2 = -\Delta t^2\frac{1}{2}gn^2 = z_n$. Therefore this method gives the exact solution.