A10101W1

# FIRST PUBLIC EXAMINATION

## Preliminary Examination in Computer Science

---

# Digital Systems, Linear Algebra, Introduction to Formal Proof

---

**TRINITY TERM 2015**

**Friday 19th June, 9:30 am – 12:30 pm**

*Candidates should answer no more than **five** questions.*
*with no more than **two** questions from each section.*
*Please start the answer to each question on a new page.*

Answers to Questions 1–3, Questions 4–6 and Questions 7–8 should be submitted in separate booklets. Please submit an empty booklet for each section of the paper if no question from that section has been attempted.

Do **not** turn over until told that you may do so.

# Linear algebra

## Question 1

Assuming there are only three bus stations in Oxford; Central, Headington, and Botley. Each day, 30% of buses from Central go to Botley, 10% to Headington and the remaining 60% stay in Central. Moreover, 10% of buses in Headington go to Botley, 10% to Central and the remaining 80% stay in Headington. Finally, 10% of Botley buses go to Central, 20% to Headington and 70% stay there. The transition matrix is given by

$$P = \begin{bmatrix} 0.6 & 0.1 & C \\ A & 0.8 & 0.2 \\ 0.3 & B & 0.7 \end{bmatrix}.$$

(a) What are the the values $A, B$ and $C$? (2 marks)

(b) What is meant by an eigenvalue and eigenvector pair of a square matrix? (2 marks)

(c) Initially 40% of buses are at Central, 40% at Headington and 20% at Botley station. Calculate the distribution over the following two days. (2 marks)

(d) Find the eigenvalue characteristic equation, $p(\lambda)$, of $P$. (4 marks)

(e) Show that one of the eigenvalues of $P$ is equal to one and find its corresponding eigenvector, $\mathbf{v}$, such that $\mathbf{v}$ is non–negative and the entries sum to one. (5 marks)

(f) By finding the remaining two eigenvalues what can be said about the convergence of the distribution vector? Justify your answer. (5 marks)

## Question 2

(a) Define the rank, row space, null space and column space for an $m$ by $n$ matrix $A$. (3 marks)

(b) Let

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 1 & 2 & 4 & 1 \\ -1 & -2 & 6 & -7 \end{bmatrix}.$$

Find a basis for the row space of $A$. What is the rank of $A$? (5 marks)

(c) Give a basis for the column space of $A$. (3 marks)

(d) Find a basis for the null space of $A$ and give its dimension. (4 marks)

(e) Find all possible real solutions to the problem

$$A\mathbf{x} = \begin{bmatrix} 2 \\ 7 \\ 3 \end{bmatrix}.$$

(5 marks)

## Question 3

(a) Show that the product of two upper triangular matrices is an upper triangular matrix. (4 marks)

(b) It is known that the product of two unit lower triangular matrices is also a unit lower triangular matrix, and that the inverses, if they exist, of both upper and unit lower triangular matrices are similarly upper and unit lower triangular matrices, respectively. Show that for any invertible matrix $A$ the $LU$ factorisation, with $L$ unit lower and $U$ upper triangular, is unique. (6 marks)

(c) Let

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}.$$

Use elementary matrix operations to row reduce $A$ to an upper triangular matrix $U$. Hence, state the $LU$ factorisation of $A$, where $L$ is a unit lower triangular matrix. (8 marks)

(d) What is the determinant of $A$? (2 marks)

# Digital systems

## Question 4

An up-down 3-bit binary counter is required. There is one control input ($A$) and a clock ($CLK$). The outputs are to be labelled $B0$, $B1$ and $B2$. If $A = 1$ then the counter counts up every clock cycle, if $A = 0$ it counts down.

(a) Design this counter in terms of AND, OR and XOR gates, inverters and D-type flip-flops. (15 marks)

(b) Suppose the propagation delay of an inverter is 1ns, that of a 2-input AND or OR gate is 3ns, and the propagation delay of a 2-input XOR gate is 2ns.

If $k$-input AND or OR gates were built using the same CMOS technology, what propagation delays would you estimate them to have?

Calculate the longest propagation delay that occurs in the combinational part of your design of the up-down 3-bit binary counter from part (a). (5 marks)

## Question 5

The IEEE-754 half-precision floating point format has a sign bit, a 5-bit exponent and a 10-bit significand packed in a 16-bit word. The exponent is represented with a bias of 15. The exponent value 0 is reserved for zero and denormalised values, and the exponent value 31 is reserved for exceptional values.

(a) What are the normalised representations of the following numbers: $\frac{1}{2}$, $\frac{5}{8}$, $-\frac{3}{16}$ and $15\frac{1}{2}$? Give your answers in binary. (6 marks)

(b) Use the representations you gave in part (a) to show the steps involved in performing the addition $\frac{5}{8} + 15\frac{1}{2}$. (4 marks)

(c) Determine the smallest and largest positive numbers that can be represented in normalized form in half-precision format. What is the smallest strictly positive number that can be represented as a denormalized value? (5 marks)

(d) Given the bit pattern:

1011 1101 0111 0000

what does it represent, assuming that it is:

- a two's complement integer;
- an unsigned integer;
- a half-precision floating point number.

If you find it convenient, you may express the results as sums or differences of powers of two instead of as numeric values. (5 marks)

## Question 6

(a) A certain microprocessor has only *zero-address* instructions, apart from one special instruction for loading a constant value.

Explain why this machine almost certainly has a stack-based architecture.

Outline the effect of executing the `add` instruction for performing addition, and that of executing the instruction for loading a constant. (4 marks)

(b) What (zero-address) instructions would such a machine have in order to make it possible to use variables with values stored in locations that have known fixed addresses? Show how to implement

```
x := y
```

if `x` and `y` are variables stored at known addresses. (4 marks)

(c) Show how this machine could use variables whose locations are stored in other variables, or whose locations are calculated by evaluating expressions. In particular explain a sequence of instructions which might be used to implement the assignment

```
a[i] := a[i] + 1
```

(4 marks)

(d) Suppose that $depth(e)$ is the amount of stack space needed to compute the value of the expression $e$. How much stack space would be needed to compute $e + f$?

Show how to exploit the commutativity of addition to minimize the space required to compute an expression involving many additions. (4 marks)

(e) Some operations, such as subtraction, are not commutative.

Suggest two possible ways of extending the instruction set of the machine to allow the same space-minimization technique to apply to expressions involving non-commutative operators.

What are the relative advantages and disadvantages of your proposed extensions?

(4 marks)

## Introduction to formal proof

*When asked for a proof in this section of the paper you are expected, unless the question states otherwise, to give a formal proof using the proof rules in Figures 1, 2, and 3. Proofs may be presented in appropriately-indented and decorated linear form, as natural deduction inference trees, or as sequent trees. When it is appropriate you may use a result proven in part of a question in the proof of a subsequent part.*

### Question 7

(a) (i) Briefly explain the Natural Deduction rules $\wedge$-intro, $\vee$-elim and $\rightarrow$-intro. What is the significance of the rectangular boxes that surround some subproofs in the latter two rules? (2 marks)

(ii) Give a *sequent* rule for $\vee$-elim and explain briefly the relationship between this and the corresponding Natural Deduction rule. (2 marks)

(b) (i) Prove $E \rightarrow F \rightarrow G \rightarrow H \vdash G \rightarrow F \rightarrow E \rightarrow H$ (2 marks)

(ii) Prove $E \rightarrow F \rightarrow G \rightarrow H \vdash ((E \wedge F) \wedge G) \rightarrow H$ (2 marks)

(c) (i) Prove $\neg P \wedge \neg Q \vdash \neg(P \vee Q)$ (3 marks)

(ii) Prove $\neg(P \vee Q) \vdash \neg P \wedge \neg Q$. (3 marks)

(iii) Explain briefly what is meant by the form of argument known as "proof by contradiction" (also as "*reductio ad absurdum*") and *state* an additional proof rule (in either natural deduction or sequent form) that embodies it. *You need not justify the rule you state.* (1 mark)

(iv) Prove $\neg(P \wedge Q) \vdash \neg P \vee \neg Q$ (5 marks)

### Question 8

In this question $R$ is a unary relation symbol.

(a) (i) Explain what it means for a variable to be *fresh* in a context, and give a brief informal justification for the sequent rule for $\forall$-introduction. (2 marks)

(ii) Prove that $\neg\exists x \cdot R(x) \vdash \forall y \cdot \neg R(y)$ (4 marks)

(iii) Prove that $\neg\exists x \cdot \neg R(x) \vdash \forall y \cdot R(y)$ (4 marks)

(b) Prove that $\exists x \cdot R(x) \vdash \neg\forall y \cdot \neg R(y)$ (4 marks)

(c) Let $c_0$, $c_1$ be constant symbols. Prove

$$\forall x \cdot (x = c_0 \vee x = c_1) \vdash \neg R(c_1) \rightarrow (\exists x \cdot R(x)) \rightarrow R(c_0)$$

(6 marks)

$$\frac{\phi \qquad \psi}{\phi \land \psi}\ \land\text{-intro} \qquad \frac{\phi \land \psi}{\phi}\ \land\text{-elim-L} \qquad \frac{\phi \land \psi}{\psi}\ \land\text{-elim-R}$$

$$\frac{\phi}{\phi \lor \psi}\ \lor\text{-intro-L} \qquad \frac{\psi}{\phi \lor \psi}\ \lor\text{-intro-R} \qquad \frac{(\phi \lor \psi) \quad \boxed{\begin{array}{c}\phi\\ \vdots\\ \kappa\end{array}} \quad \boxed{\begin{array}{c}\psi\\ \vdots\\ \kappa\end{array}}}{\kappa}\ \lor\text{-elim}$$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \to \psi}\ \to\text{-intro} \qquad \frac{\phi \quad \phi \to \psi}{\psi}\ \to\text{-elim}$$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ \neg\text{-intro} \qquad \frac{\phi \quad \neg\phi}{\bot}\ \neg\text{-elim} \qquad \frac{\neg\neg\phi}{\phi}\ \neg\neg\text{-elim}$$

$$\frac{\bot}{\phi}\ \bot\text{-elim}$$

$$\frac{\phi \to \psi \quad \psi \to \phi}{\phi \leftrightarrow \psi}\ \leftrightarrow\text{-intro} \qquad \frac{\phi \leftrightarrow \psi}{\phi \to \psi}\ \leftrightarrow\text{-elim-R} \qquad \frac{\phi \leftrightarrow \psi}{\psi \to \phi}\ \leftrightarrow\text{-elim-L}$$

Figure 1: Natural Deduction rules for the logical connectives

$$\frac{\Gamma \vdash \forall x \cdot \phi(x)}{\Gamma \vdash \phi(T)}\ \forall\text{-elim}\ (T \text{ free for } x \text{ in } \phi(x)) \qquad \frac{\Gamma \vdash \phi(v)}{\Gamma \vdash \forall x \cdot \phi(x)}\ \forall\text{-intro}\ \ (v \text{ is fresh})$$

$$\frac{\Gamma \vdash \phi(T)}{\Gamma \vdash \exists x \cdot \phi(x)}\ \exists\text{-intro}\ (T \text{ free for } x \text{ in } \phi(x)) \qquad \frac{\Gamma, \phi(v) \vdash \kappa}{\Gamma, \exists x \cdot \phi(x) \vdash \kappa}\ \exists\text{-elim}\ \ (v \text{ is fresh})$$

Figure 2: Sequent rules for the quantifiers

$$\frac{}{T = T}\ =\text{-intro} \qquad \frac{T_1 = T_2 \qquad \phi(T_1)}{\phi(T_2)}\ =\text{-elim}$$

Figure 3: Natural Deduction Rules for Equality