Discrete Mathematics (Sheet #2)

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1. (a) $\text{Im}(f) = \mathbb{R}$ so the function is onto. Function is also 1-1 because $f(x) = f(y) \implies 2x+3 = 2y+3 \implies x = y$. $f^{-1}(x) = \frac{x-3}{2}$.
   (b) $f(x) = x(x-1)^2$.
   $\text{Im}(f) = \mathbb{R}$ so the function is onto. $f(0) = f(1) = 0$ so the function is not 1-1.
   (c) $f(\frac{\pi}{4}) = \frac{1}{2}, f(\frac{3\pi}{4}) = -\frac{1}{2} \implies \text{Im}(f) = [-1/2, 1/2]$ so the function is onto. $f(2\pi + k) = f(k)$ so the function is not 1-1.
   (d) $f^{-1}(x) = \frac{1}{1-|x|}$. Since f has an inverse, it is both 1-1 and onto.

2. (a) $\pi/2$
   (b) $\text{Im}(f) = \mathbb{R}$. f is a bijection.
   (c) $g : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
   (d) $h : \mathbb{R} \to (-1, 1)$ where $h(x) = \frac{2g(x)}{\pi}$.

3. $f(x, y) = (x - 2y)(x + 2y)$
   Since $f(1, 2) = f(2, 4) = 0$, f is not 1-1.
   Suppose f is onto. Then $\exists x_0, y_0 \in \mathbb{N}$ such that $f(x_0, y_0) = 7 \iff (x_0 - 2y_0)(x_0 + 2y_0) = 7$. Since 7 is a prime number and $x_0 - 2y_0 < x_0 + 2y_0$, we get $x_0 - 2y_0 = 1, x_0 + 2y_0 = 7$ which has no solutions in $\mathbb{N}^2$. So f is not onto.

4. (a) 2^5 functions in total, 0 1-1 functions and $2^5 - 2$ onto functions.
   (b) n^m functions in total, $m!\binom{n}{m}$ 1-1 functions.

5. (a) $\displaystyle\sum_{k=0}^{8} 9 - k = 45$
   (b) $\displaystyle\sum_{k=5}^{8} 9 - k = 10$
   (c) $\displaystyle 3\sum_{k=5}^{8} 9 - k = 30$
   (d) $\displaystyle\sum_{k=0}^{14}(14 - k) - 3\sum_{k=10}^{14}(14 - k) = 105 - 3 \cdot 10 = 75$

6. There are $\lfloor \frac{N}{k} \rfloor$ positive integers up to N divisble by k. Let a good number be a positive integer divisible by one or more of 5,6 or 8.
   Good numbers less than 1000: $\lfloor \frac{999}{5} \rfloor + \lfloor \frac{999}{6} \rfloor + \lfloor \frac{999}{8} \rfloor - \lfloor \frac{999}{24} \rfloor - \lfloor \frac{999}{30} \rfloor - \lfloor \frac{999}{40} \rfloor + \lfloor \frac{999}{120} \rfloor = 199 + 166 + 124 - 41 - 33 - 24 + 8 = 399$.

Good numbers less than 100: $\lfloor \frac{99}{5} \rfloor + \lfloor \frac{99}{6} \rfloor + \lfloor \frac{99}{8} \rfloor - \lfloor \frac{99}{24} \rfloor - \lfloor \frac{99}{30} \rfloor - \lfloor \frac{99}{40} \rfloor + \lfloor \frac{99}{120} \rfloor = 19 + 16 + 12 - 4 - 3 - 2 + 0 = 38$.

Good numbers with exactly three digits: $399 - 38 = 361$.

7. Multinomial coefficients count the ways of placing a objects into b bins, such that k_i objects are placed in the ith bin.

   For $a = mn, b = m, k_i = n$, the multinomial coefficient is

$$\binom{mn}{\underbrace{m\ m\ m\ \dots\ m}_{n \text{ times}}} = \frac{(mn)!}{\underbrace{m!m!m!\dots m!}_{n \text{ times}}} = \frac{(mn)!}{(m!)^n}$$

   Since the number of ways to do something is an integer, the numerator must $(mn)!$ be divisible by the denominator $(m!)^n$.