Discrete Mathematics (Sheet #3)

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1. (a) $<$ is antisymmetric, irreflexive, transitive, serial and not symmetric.
   (b) $R = \{(a, b), (b, a), (a, c), (a, a)\}$ is not symmetric, not antisymmetric, not reflexive, not irreflexive.
   (c) $R = \{(a, b), (b, c), (c, b), (c, a), (b, a), (a, a), (b, b), (c, c)\}$ is not symmetric.
   $R^+ = \{(a, b), (b, c), (a, c), (c, b), (c, a), (b, a), (a, a), (b, b), (c, c)\}$ is symmetric.

2. On $\{0, 1\}$: $2^4 = 16$ relations, $2^2 = 4$ reflexive relations, $2^3 = 8$ symmetric relations. $2^4 - 1 = 15$ transitive relations.

   There are $\frac{n(n-1)}{2}$ possible elements in the relation, therefore there are $2^{\frac{n(n-1)}{2}}$ antisymmetric relations.

3. (a) $n = 2^0 n \implies n \sim n$. $n = 2^k m \implies m = 2^{-k} n$, so $m \sim n \implies n \sim m$, $a = 2^k b, b = 2^t c \implies a = 2^{k+t} c$, so $a \sim b, b \sim c \implies a \sim c$. So $\sim$ is an equivalence relation.
   (b) Equivalence classes are $\{1, 2, 4, 8, 16\}, \{3, 6, 12\}, \{5, 10\}, \{7, 14\}, \{9\}, \{11\}, \{13\}, \{15\}$.

4. (a) Base case: $F_0 = F_2 - 1$.
   Suppose $F_0 + F_1 + \ldots + F_n = F_{n+2} - 1$. Then $F_0 + F_1 + \ldots + F_n + F_{n+1} = F_{n+1} + F_{n+2} - 1 = F_{n+3} - 1$.
   (b) Base case: $0 = 0F_2 - F_3 + 2 = 0 - 2 + 2$.
   Suppose $0F_0 + 1F_1 + \ldots + nF_n = nF_{n+2} - F_{n+3} + 2$. Then $0F_0 + 1F_1 + \ldots + nF_n + (n+1)F_{n+1} = (n+1)F_{n+1} + nF_{n+2} - F_{n+3} + 2 = (n+1)F_{n+3} - F_{n+2} - F_{n+3} + 2 = nF_{n+3} - F_{n+2} + 2$.

5. The generalised statement: $\sum_{i=1}^{n} \prod_{k=0}^{m-1} (i + k) = \frac{\prod_{k=0}^{m}(n+k)}{(m+1)}$

   Base case: $\prod_{k=0}^{m-1} (1 + k) = \frac{\prod_{k=0}^{m}(1+k)}{(m+1)}$.

   Suppose the statement is true for n. Then $\displaystyle\sum_{i=1}^{n+1} \prod_{k=0}^{m-1} (i + k) = \frac{\prod_{k=0}^{m}(n + k)}{m + 1} + \prod_{k=0}^{m-1} (i + n + 1) =$

   $$\frac{n \prod_{k=1}^{m}(n + k) + (m + 1) \prod_{k=1}^{m}(n + k)}{m + 1} =$$
   $$\frac{\prod_{k=0}^{m-1}(n + k + 1)(n + m + 1)}{m + 1} = \frac{\prod_{k=0}^{m}(n + k + 1)}{m + 1}$$

6. $a_1 = 1, a_2 = 2a_1 = 2, a_3 = 3a_1 = 3, a_4 = 4a_2 = 8, a_5 = 5a_2 = 10, a_6 = 6a_3 = 18$.

   Suppose the statement $a_n \le n^{\log_2 n}$ is true for $n < k$.

   $a_k = k a_{\lfloor k/2 \rfloor} \le k k^{\log_2 \lfloor k/2 \rfloor} \le k^{1 + \log_2 k/2} = k^{\log_2 2k/2} = k^{\log_2 k}$. So the statement is also true for k.

7. (a) Take a binary tree with n nodes in total, k nodes in the left subtree of the root and $n - k - 1$ nodes in the right subtree. There are b_k ways to arrange the k nodes in the left subtree and b_{n-k-1} ways to arrange the $n - k - 1$ nodes in the right subtree. Thus, there are $b_k b_{n-k-1}$ such trees.
   By varying k from 0 to $n - 1$ we get $b_n = b_0 b_{n-1} + b_1 b_{n-2} + \ldots + b_{n-2} b_1 + b_{n-1} b_0$.

(b) $b_0 = 1, b_1 = 1$. $b_2 = b_0b_1+b_1b_2 = 2$. $b_3 = b_0b_2+b_1b_1+b_2b_0 = 5$. $b_4 = b_0b_3+b_1b_2+b_2b_1+b_3b_0 = 14$. $b_5 = b_0b_4 + b_1b_3 + b_2b_2 + b_3b_1 + b_4b_0 = 28 + 10 + 4 = 42$.

This agrees with the typical formula for Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$ because $C_5 = \frac{1}{6}\binom{10}{5} = \frac{252}{6} = 42 = b_5$.