Discrete Mathematics (Sheet #5)

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1. (a) Any two distinct elements of the chain must have different sizes. There are 7 possible sizes (from 0 to 6), so a chan can have at most seven elements.

   (b) We have 7 chains with pairwise different sizes, and 7 possible sizes. Let B_i be the chain with $i-1$ elements.

   If we fix B_i, then $B_{i+1} = B_i \cup \{a\}$, where $a \notin B_i$. So there are $n-i$ possibilities for B_{i+1}. As there is only one possible B_0 (namely $\varnothing$), we have n possibilities for B_1, $n-1$ possibilities for B_2 for each B_1 and so on. In total, there are $n! = 6! = 720$ chains.

   (c) It is the antichain made of all the elements of A with a cardinality of 3. There are $\binom{6}{3} = 20$ such elements, and any two of them will be different and have the same cardinality, so they will not be comparable.

2. (a) Preorder, as $\{1,2\} \preceq \{0,2\}$ and $\{0,2\} \preceq \{1,2\}$.

   (b) Partial order, $\sup(\{(x_n),(y_n)\}) = (z_n)$ where $z_i = \max(x_i, y_i) \forall i \in \mathbb{N}$.

   (c) Partial order, $\sup(\{2,11\})$ does not exist as $22 \notin S$.

   (d) Total order, $\sup(a,b) = \min(a,b)$.

3. POZA AICI.

   (a) $\inf((1,2),(1,3)) = (1,1)$.

   (b) $\inf((2,3),(3,2)) = (1,6)$.

   (c) $\inf((6,2),(3,3)) = (3,3)$.

4. Take any $x = a+b$ where $a \in A \wedge b \in B$. Since $a \le \sup A \wedge b \le \sup B$ we have $x \le \sup A + \sup B$.

   Suppose $\exists \varepsilon > 0$ such that $\sup A + \sup B - 2\varepsilon \ge x \forall x = a+b$ where $a \in A \wedge b \in B$. We know $\exists a' \in A \wedge b' \in B$ such that $a' > \sup A - \varepsilon \wedge b' > \sup B - \varepsilon$. But then we have $x' = a' + b' > \sup A + \sup B - 2\varepsilon$, contradicting our supposition. Therefore, $\forall y < \sup A + \sup B$ there exists an $x = a+b$ where $a \in A \wedge b \in B$ such that $y < x$.

   From the previous two paragraphs, we get $\sup A + \sup B$ is $\sup\{a+b | a \in A \wedge b \in B\}$.

5. $S \cup T = (S \setminus T) \cup (T \setminus S) \cup (S \cap T) = (S \oplus T) \cup (S \cap T)$.

   So $S \cup T = S \oplus T \iff S \cap T = \varnothing$.

6. $f(m,n) = \lfloor \frac{10^m - 1}{n} \rfloor - \lfloor \frac{10^{m-1}-1}{n} \rfloor$.

   4-digit positive integers divisible by 6 and 15: $f(4,30) = \lfloor \frac{9999}{30} \rfloor - \lfloor \frac{999}{30} \rfloor = 300$.

   4-digit positive integers divisible by 6 or 15: $f(4,6) + f(4,15) - f(4,30) = 1500 + 600 - 300 = 1800$.

   4-digit positive integers divisible by 6 or 10 or 15: $f(4,6) + f(4,10) + f(4,15) - f(4,30) - f(4,30) - f(4,30) + f(4,30) = 1500 + 900 + 600 - 600 = 2400$

7. (a) $m \equiv n \pmod{\text{n-m}}$. Therefore $m^k \equiv n^k \pmod{\text{n-m}} \forall k \in \mathbb{Z}_+$. So $n-m | n^k - m^k \forall k \in \mathbb{Z}_+$. $f(n) - f(m) = \sum_{i=0}^{k} a_i(n^i - m^i)$. Every element of this sum is a multiple of $n - m$, so the sum is a multiple of $n - m$.

   (b) Suppose $f(0) = f(3) = 0$ and $f(n) = 1$ for some n. Then $n - 0 | f(n) - f(0) \iff n | 1$, so $n = 1$. But we also have $n - 3 | f(n) - f(3) \iff n - 3 | 1$, so $n - 3 = \pm 1$, which contradicts $n = 1$.

8. (a) $C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14$.

   (b) Base case: $C_0 = 1 = \frac{1}{0+1}\binom{2 \cdot 0}{0}$.

   Suppose $C_n = \frac{1}{n+1}\binom{2n}{n}$. Then $C_{n+1} = \frac{2(2n+1)}{n+2} \frac{1}{n+1} \frac{(2n)!}{n!n!} = \frac{2(n+1)}{n+2} \frac{(2n+1)!}{(n+1)!(n+1)!} = \frac{1}{n+2} \frac{(2n+2)!}{(n+1)!(n+1)!} = \frac{1}{n+2}\binom{2n+2}{n+1}$.

   (c) $p | C_n = \frac{1}{n+1} \frac{(2n)!}{n!n!} \implies p | (2n)!$. Suppose $p \geq 2n$. Since $2n$ is not prime, we have $p > 2n$. But then no integer between 1 and $2n$ will be a multiple of p, so $(2n)! = 1 \cdot 2 \cdot 3 \cdot \ldots 2n$ will not be a multiple of p.

   (d) We have $n^2 > n + 1 \forall n \geq 4$.

   Base case: $C_4 = 14 > 7$.

   Suppose $C_n > 2n - 1$. Then $C_{n+1} = \frac{2(2n+1)}{n+2} C_n > \frac{2(2n+1)(2n-1)}{n+2} = \frac{2(4n^2-1)}{n+2} > \frac{2n^2+6(n+1)}{>} \frac{2n^2+5n+2}{n+2} = 2(n+1) - 1$.

   (e) Suppose C_n is prime for some $n \geq 4$. From (iii) we get $C_n < 2n$. From (iv) we get $C_n > 2n - 1$, contradiction.

   (f) $C_n = \frac{1}{n+1} \frac{(2n)!}{n!n!} < \frac{1}{n+1} \frac{c(2n)^{2n+\frac{1}{2}} \exp(-2n)}{(bn^{n+\frac{1}{2}} \exp(-n))^2} = \frac{1}{n+1} \frac{c2^{2n+\frac{1}{2}}}{b^2 n^{\frac{1}{2}}} = 4^n \frac{c\sqrt{2}}{b^2} \frac{1}{(n+1)n^{\frac{1}{2}}} = O(4^n n^{-3/2})$