Linear Algebra (Sheet #1)

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1. The dot product takes two vector operands and produces a scalar.
   (a) Adding a scalar ($u \cdot v$) and a vector (w) is undefined.
   (b) Applying the dot product to a vector (u) and a scalar ($v \cdot w$) is undefined.
   (c) Applying the dot product to a scalar (c) and a vector ($v + w$) is undefined.

2. $|u+v| = |u| + |v| \implies (|u+v|)^2 = (|u|+|v|)^2 \implies (u+v)^2 = u^2 + v^2 + 2|u||v| \implies 2u \cdot v = 2|u||v| \implies u \cdot v = |u||v| \implies \cos\theta = \frac{u \cdot v}{|u||v|} = 1$

   Therefore the equality holds only when the angle between the vectors is 0, in other words only when the vectors are collinear.

3.

4. Let $x_1 = p_1 + td_1$ and $x_2 = p_1 + td_2$ be the two lines and let $d_1 = [a_1, b_1]$ and $d_2 = [a_2, b_2]$. Since x_1 is parallel to d_1, and x_2 is parallel to d_2, we get:

   x_1 and x_2 are perpendicular $\iff$ d_1 and d_2 are perpendicular $\iff d_1 \cdot d_2 = 0 \iff a_1a_2 + b_1b_2 = 0 \iff a_1a_2 = -b_1b_2 \iff \frac{b_1}{a_1}\frac{b_2}{a_2} = -1$.

5. We will look at angle between the planes' normals.

   For plane $\mathcal{P}_1$, $n_1 = [4, -1, 5]$ and $|n_1| = \sqrt{16+1+25} = \sqrt{42}$.

   Planes are perpendicular if $\cos\theta = 0 \iff \frac{n_x \cdot n_y}{|n_x||n_y|} = 0 \iff n_x \cdot n_y = 0$, and parallel if $\alpha n_x = n_y$ for some $\alpha \in \mathbb{R}$.

   (a) $n_a = [2, 3, -1]$
   $n_1 \cdot n_a = 8 - 3 - 5 = 0$, so the planes are perpendicular.
   Planes

   (b) $n_b = [4, -1, 5]$
   Same normals, so the planes are parallel.

   (c) $n_c = [1, -1, -1]$
   $n_1 \cdot n_c = 4 + 1 - 5 = 0$, so the planes are perpendicular.

   (d) $n_d = [4, 6, -2]$
   $n_1 \cdot n_d = 16 - 6 - 10 = 0$, so the planes are perpendicular.

6. (a) $\left[\begin{array}{ccc|c} 2 & -3 & 1 & 9 \\ -1 & 1 & 2 & 0 \\ -1 & 1 & 4 & 4 \end{array}\right]$

   $\left[\begin{array}{ccc|c} 0 & -1 & 5 & 9 \\ -1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 4 \end{array}\right]$

   $\left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & 2 & 4 \end{array}\right]$

   $2x = 4 \implies x = 2$
   $-z + 5x = 9 \implies z = 1$
   $-y + z + 2x = 0 \implies y = 5$

(b) $\left[\begin{array}{ccccc|c} 2 & 1 & -1 & -1 & 2 & 3 \\ 0 & 1 & -2 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right]$

Free variables $x_4 = a$, $x_5 = b$.

$x_3 = 2 + b - 2a$
$x_2 = -1 - b - a + 2(2 + b - 2a) = 3 + b - 4a$
$2x_1 = 3 - 2b + a + (2 + b - 2a) - (3 + b - 4a) = 2 - 2b + 3a \implies x_1 = 1 - b + 1.5a$

(c) Adding first two equations yields the third, so we can ignore the third equation.

$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -2 \\ 2 & 1 & 3 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|c} 1 & -2 & -1 & -2 \\ 0 & 5 & 5 & 5 \end{array}\right]$

Free variable $x_3 = a$.

$x_2 + x_3 = 1 \implies x_2 = 1 - a$
$x_1 - 2x_2 - x_3 = -2 \implies x_1 = 2 - 2a - a - 2 = -3a$

7. $\left[\begin{array}{ccc|c} 3 & 2 & 1 & -1 \\ 2 & -1 & 4 & 5 \end{array}\right]$

$\left[\begin{array}{ccc|c} 6 & 4 & 2 & -2 \\ 6 & -3 & 12 & 15 \end{array}\right]$

$\left[\begin{array}{ccc|c} 6 & 4 & 2 & -2 \\ 0 & -7 & 10 & 17 \end{array}\right]$

Free variable $z = a$.

$-7y + 10a = 17 \implies y = \frac{10a-17}{7}$
$3x + 2y + z = -1 \implies x = -\frac{\frac{20a-34}{7}+a+1}{3} = -\frac{20a-34+7a+7}{21} = \frac{27a-27}{21}$

Intersection is $(\frac{9}{7}(1-a), \frac{10a-17}{7}, a)$

8. Free variable $u = a$.

$w = 2 - a$
$z = 4 - 2 = 2$
$v = 6 - 4 = 2$

If $u = -1$, then the intersection is $(-1, 2, 3, 2)$.

If we include the plane $u + w = 3$, there is no solution.

Applications:

1. $c \cdot x + 4 = 0 \implies 10 + 4 + 2 + 2 + 6 + 4 + 10 + 6 + 14 + 8 + 18 + 0 + 8 + 3 + 4 + d + 4 = 0 \implies d = 7$

   For $x' = 5421\ 3456\ 7890\ 4327$ we get

   $c \cdot x' + 4 = 10 + 4 + 4 + 1 + 6 + 4 + 10 + 6 + 14 + 8 + 18 + 0 + 8 + 3 + 4 + 7 + 4 = 1 \neq 0$

2. $0 * 10 + 5 * 9 + 3 * 8 + 4 * 7 + 4 * 6 + 2 * 5 + 2 * 4 + 0 * 3 + 0 * 2 + x = 0 \implies x = 4$

   $0 * 10 + 8 * 9 + 3 * 8 + 7 * 7 + 0 * 6 + 9 * 5 + 9 * 4 + 0 * 3 + 2 * 2 + 6 = 5 \neq 0$

   The correct ISBN is $[0, 3, 8, 7, 0, 9, 9, 0, 2, 6]$

3. (a) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 20 \\ 1 & 1 & 0 & 0 & 25 \\ 0 & 1 & 1 & 0 & 30 \\ 0 & 0 & 1 & 1 & 25 \end{array}\right]$

   (b) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 20 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 1 & 1 & 25 \end{array}\right]$

   Free variable $f_4 = a$

   $f_3 = 25 - a$
   $f_2 - a = 5 \implies f_2 = 5 + a$
   $f_1 + a = 20 \implies f_a = 20 - a$

(c) $f_2 = 5 + a = 10 \implies a = 5$
$f_4 = a = 5$
$f_1 = 20 - a = 15$
$f_3 = 25 - a = 20$
The flows are $(15, 10, 20, 5)$

(d) $0 \leq a \leq 20$. Therefore:
$0 \leq f_1 \leq 20$
$5 \leq f_2 \leq 25$
$5 \leq f_3 \leq 25$
$0 \leq f_4 \leq 20$

4. $$\left[\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 2 & 1 & 14 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 2 & 1 & 14 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 2 & 1 & 14 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 2 & 1 & 14 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 6 & -3 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 2 & 1 & 14 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & -6 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 2 & 1 & 14 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -21 & -126 \\ 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 14 \\ 0 & 0 & 0 & 0 & 0 & -21 & -126 \end{array}\right]$$

$I_5 = 6A$
$I_4 = 4A$
$I_3 + I_4 - I_5 = 0 \implies I_3 = 2A$
$-I_2 + -I_3 + 3I_4 - I_5 = 0 \implies I_2 = 4A$

$I_1 + I_3 - 2I_4 = 0 \implies I_1 = 6A$
$I - I_1 - I_4 = 0 \implies I = 10A$

5. $\dfrac{14V}{10A} = 1.4$ ohms