Linear Algebra (Sheet #3)

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1. (a) Matrix addition is closed, commutative, associative, has a zero (the 0 matrix), and all matrices have an additive inverse (the matrix with all elements negated).

   Matrix scalar multiplication is closed, is distributive, $\alpha(\beta M) = (\alpha\beta)M$ and there exists a multiplicative identity (I).

   Therefore this is a vector space.

   (b) Polynomial addition is closed, commutative, associative, has a zero (0), and all polynomials have an additive inverse (the polynomial with all coefficients negated).

   Polynomial scalar multiplication is closed, $\alpha(\beta P) = (\alpha\beta)P$, and there exists a multiplicative identity (1).

   Therefore this is a vector space.

   (c) Not a vector space because it has no multiplicative identity. For example, no scalar α exists such that $\alpha[1,1]^T = [1,1]^T$.

2. Since all of the following spaces are subsets of a vector space, we just need to check whether addition and scalar multiplication are closed.

   (a) Closed: Adding two symmetric matrices yields a symmetric matrix, and multiplying a symmetric matrix by a scalar still yields a symmetric matrix.

   (b) Closed: Adding two polynomials of degree $n-1$ or less or multiplying one such polynomial by a scalar will yield another polynomial of degree $n-1$ or less.

   (c) Not closed, for example: $[1,1]^T + [2,4]^T = [3,5]^T$, which is not a member of this subset.

   (d) Closed: $[3a, a, -2a]^T + [3b, b, -2b]^T = [3(a+b), (a+b), -2(a+b)]^T$ and $\alpha[3a, a, -2a]^T = [(3\alpha)a, \alpha a, -2\alpha a]^T$.

   (e) Not closed, for example $[3+1, 1, -2]^T + [1, 0, 0]^T = [5, 1, -2]^T$ which is not a member of this subset.

3. Each element of $\mathcal{C}(A)$ is an element of $\mathbb{R}^m$.

   $\mathbb{R}^m$ is a vector space.

   We just need to prove that $\mathcal{C}(A)$ is closed. By the definition of $\mathcal{C}(A)$, it contains all vectors of the form $a_1\vec{c_1} + a_2\vec{c_2} + \ldots + a_m\vec{c_m}$, where $\vec{c_i}$ are some of the columns of A.

   Let $v_1 = \sum_{i=1}^{m} a_i\vec{c_i}, v_2 = \sum_{i=1}^{m} b_i\vec{c_i}$ be two elements of $\mathcal{C}(A)$ and $\alpha \in \mathbb{R}$.

   Then $v_1 + v_2 = \sum_{i=1}^{m}(a_i + b_i)\vec{c_i}$ and $\alpha v_1 = \sum_{i=1}^{m}(\alpha a_i)\vec{c_i}$ are both elements of $\mathcal{C}(A)$, so it is closed under both summation and scalar multiplication. Therefore $\mathcal{C}(A)$ is a subspace of $\mathbb{R}^m$

   By the fundamental theorem of invertible matrices, A is invertible implies that the column vectors of A are linearly independent and span $\mathbb{R}^n$. Therefore $\mathcal{C}(A) = span(\{c_1, c_2, \ldots, c_n\}) = \mathbb{R}^n$

4. Analogously to the previous problem, we just need to prove that $\mathcal{N}(A)$ is closed.

   If $x, y \in \mathcal{N}(A)$ and $\alpha \in \mathbb{R}$ then, by the definition of the nullspace, $Ax = Ay = 0$. But then $A(x+y) = Ax + Ay = 0$ and $A(\alpha x) = \alpha Ax = 0$. So $\mathcal{N}(A)$ is closed under both summation and scalar multiplication. Therefore it is a subspace of $\mathbb{R}^n$.

   If A is invertible then $Ax = 0$ has only the trivial solution, so $\mathcal{N}(A) = 0$.

5. $\begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 2 & -3 \\ 2 & 4 & 8 & 5 \end{bmatrix}$

   $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 2 & 6 & 1 \end{bmatrix}$

   $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

   We get $x_4 = 0$
   $x_3 = a$
   $x_2 + 3x_3 - x_4 = 0 \implies x_2 = -3a$
   $x_1 + x_2 + x_3 + 2x_4 = 0 \implies x_1 = 3a - a = 2a$

   The nullspace is all vectors of the form $[2a, -3a, a, 0]^T$ and $\text{rank}(A) = 3$.

6. $\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & -4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$

   $\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 5 & -14 & -9 & 0 \\ 0 & 10 & -28 & -18 & 0 \end{bmatrix}$

   $\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 0 & -29 & 1 & 0 \\ 0 & 0 & 2 & 2 & 0 \end{bmatrix}$

   $\begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 30 & 0 \end{bmatrix}$

   Leading coefficients occur in columns 1,2,3,4 so a basis for the column space is $\{c_1, c_2, c_3, c_4\}$. Since $\dim(\mathcal{R}(A)) = \dim(\mathcal{C}(A)) = 4$ and there are only 4 rows, a basis for the row space is $\{r_1, r_2, r_3, r_4\}$, where r_i is the ith row of the matrix.

   Solving $Ax = 0$ we get $x_4 = 0, x_3 = 0, x_2 = 0, x_1 = -2a, x_5 = a$. So a basis for the nullspace is $[-2, 0, 0, 0, 1]^T$. We have $[-2, 0, 0, 0, 1]^T b = 0$, so $b = [x, y, z, t, 2x]^T$.

7. (a) $\begin{bmatrix}1 & 2 & 2\\ 2 & 4 & 5\end{bmatrix}$

$\begin{bmatrix}1 & 2 & 2\\ 0 & 0 & 1\end{bmatrix}$

We get $x_n = [-2a, a, 0]^T$. * A particular solution is $x_p = [-3, 0, 2]^T$. Therefore $x = [-3-2a, a, 2]^T$ are the solutions.

(b) $x_n = [-2a-2b, a, b]$. After gauss elimination we get $0 = 2$ on the second line, so there are no solutions.

8. (a) No, because any basis for $\mathcal{P}^2$ must have the same dimension and the basis $\{1, x, x^2\}$ has dimension 3.

(b) No, because any basis for M_{22} must have the same dimension and the basis $\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$ has dimension 4.

(c) Yes. $ax^2 + bx + c = \frac{1}{2}((-a+b+c)(1+x) + (a+b-c)(x+x^2) + (a-b+c)(1+x^2))$.

9. (a) $T([x,y]^T) + T([u,v]^T) = [x+y, x-y]^T + [u+v, u-v]^T = [(x+u)+(y+v), (x+u)-(u+v)]^T = T([x+u, y+v]^T)$

Therefore T is linear.

(b) $T([x,y]^T) + T([u,v]^T) = [-y, x+2y, 3x-4y]^T + [-v, u+2v, 3u-4v]^T = [-(y+v), (x+u)+2(y+v), 3(x+u)-4(y+v)]^T = T([x+u, y+v]^T)$

Therefore T is linear.

(c) $T([x,y]^T) + T([u,v]^T) = [x+1, y-1]^T + [u+1, v-1]^T = [x+u+2, y+v-2]^T \neq [(x+u)+1, (y+v)-1]^T = T([x+u, y+v]^T)$

Therefore T is not linear.

(d) $T([x,y]^T) + T([u,v]^T) = [xy, x+y]^T + [uv, u+v]^T = [xy+uv, x+u+y+v]^T \neq [(x+u)(y+v), x+u+y+v]^T = T([x+u, y+v]^T)$

Therefore T is not linear.

(e) $T([x,y,z]^T) + T([u,v,w]^T) = [x-y+z, 2x+y-3z]^T + [u-v+w, 2u+v-3w]^T = [(x+u)-(y+v)+(z+w), 2(x+u)+(y+v)-3(z+w)]^T = T([(x+u), (y+v), (z+w)]^T)$

Therefore T is linear.

10. (a) $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$

(b) $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{bmatrix}$

(c) $\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$

(d) $\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta_3 & -\sin\theta_3\\ 0 & \sin\theta_3 & \cos\theta_3\end{bmatrix} \begin{bmatrix}\cos\theta_2 & 0 & \sin\theta_2\\ 0 & 1 & 0\\ -\sin\theta_2 & 0 & \cos\theta_2\end{bmatrix} \begin{bmatrix}\cos\theta_1 & -\sin\theta_1 & 0\\ \sin\theta_1 & \cos\theta_1 & 0\\ 0 & 0 & 1\end{bmatrix}$

11. (a) $T(1,0) = T(2,0) = (0,0)$, so the transformation is not injective.

(b) The transformation is not surjective, for example $\nexists x, y$ such that $T(x,y) = (0,1,2)$.

(c) $T(1,0) = T(1,1) = 1$, so the transformation is not injective.

Applications

1. (a) $M_1^2 = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

   $M_2^2 = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

   (b) $M_i(x, y)$ is the number of ways of length 2 from node x to node y.

   (c) $M = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

   (d) After one step, B, C, D will know the rumour. After the second step A, B, C, D, E will know the rumour. So two steps are needed.

   (e) After one step, A, C, E will know the rumour. After the second step A, B, C, E will know the rumour. After the third step, A, B, C, D, E will know the rumour. So three steps are needed.

2. (a) $G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

   (b) $Gx = [0, 1, 0, 1, 0, 1, 0]^T$.

   (c) $P(Gx) = [4, 2, 4]^T = 0$

   (d) $P([0, 1, 1, 1, 0, 1, 0]^T) = [0, 1, 1]^T$. This is column 3 of P and the third element was changed.

3. (a) $C[i, j]$ is 1 if and only if there is a line between point i and point j. To plot the house, lines will be drawn between every two points i and j where $C[i, j] = 1$.

   (b) $TV = \begin{bmatrix} 0 & 0 & -2 & -3 & -2 \\ 0 & 2 & 2 & 1 & 0 \end{bmatrix}$

   TV is V rotated 90 around the origin.

   (c) $V' = \begin{bmatrix} 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

   $T' = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

   (d) $T' = \begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$