Linear Algebra (Sheet #4)

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1. (a) $3+2=5$

   (b) $1\begin{vmatrix}3 & -3\\5 & 1\end{vmatrix}+1\begin{vmatrix}2 & -3\\4 & 1\end{vmatrix}+2\begin{vmatrix}2 & 3\\4 & 5\end{vmatrix}=3+15+2+12+20-24=28$

   (c) $1\begin{vmatrix}6 & -1\\5 & 1\end{vmatrix}+1\begin{vmatrix}3 & -1\\4 & 1\end{vmatrix}+2\begin{vmatrix}3 & 6\\4 & 5\end{vmatrix}=6+5+3+4+30-48=0$

   (d) $3\begin{vmatrix}1 & -3\\0 & 1\end{vmatrix}-2\begin{vmatrix}2 & -3\\4 & 1\end{vmatrix}+1\begin{vmatrix}2 & 1\\4 & 0\end{vmatrix}=3-4-24-4=-29$

   (e) $2\begin{vmatrix}3 & 2 & 1\\2 & 1 & -3\\4 & 0 & 1\end{vmatrix}-0+0-0=-58$

   (f) $2\begin{vmatrix}3 & 2 & 1\\0 & 4 & 3\\0 & 0 & 1\end{vmatrix}=2\cdot 3\begin{vmatrix}4 & 3\\0 & 1\end{vmatrix}=24$

2. (a) $\begin{vmatrix}1 & 2\\0 & 5\end{vmatrix}=5$

   (b) $\begin{vmatrix}1 & -1 & 2\\0 & 5 & -7\\0 & 9 & -7\end{vmatrix}=\begin{vmatrix}1 & -1 & 2\\0 & 5 & -7\\0 & 0 & \frac{4}{5}7\end{vmatrix}=28$

   (c) $\begin{vmatrix}1 & -1 & 2\\0 & 9 & -7\\0 & 9 & -7\end{vmatrix}=0$

   (d) $\begin{vmatrix}3 & 2 & 1\\0 & -\frac{1}{3} & -\frac{11}{3}\\0 & \frac{8}{3} & -\frac{1}{3}\end{vmatrix}=\begin{vmatrix}3 & 2 & 1\\0 & -\frac{1}{3} & -\frac{11}{3}\\0 & 0 & \frac{87}{3}\end{vmatrix}=-29$

   (e) $\begin{vmatrix}2 & 1 & -2 & 0\\0 & 3 & 2 & 1\\0 & 0 & -\frac{1}{3} & -\frac{11}{3}\\0 & 0 & 0 & \frac{87}{3}\end{vmatrix}=-58$

   (f) $2\cdot 3\cdot 4\cdot 1=24$

3. $Q^TQ=I \implies \det(Q^T)\det(Q)=\det(I) \implies \det(Q)^2=1 \implies \det(Q)=\pm 1$

4. $(4-\lambda)(3-\lambda)+2=\lambda^2-7\lambda+14; \Delta=49-56=-7.$
   $\lambda_{1,2}=\frac{7-\sqrt{7}i}{2}.$

5. $\det(A^T) = \det(A)$. Every column of A^T sums up to 0. If we replace the last row with the sum of all rows, the determinant stays the same. But then the matrix has a row of zeroes, so the determinant is 0.

   If every row of A sums up to 1, then every row of $A-I$ will sum up to 0, and the same reasoning applies.

   Example: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has determinant -1.

6. $\det(AB) = \det(-BA) \implies \det(A)\det(B) = (-1)^n \det(A)\det(B)$. For odd n, this statement implies $\det(A)\det(B) = 0$, so at least one of the matrices must be singular. For even n, this statement is true for any two matrices.

7. $q_1 = \dfrac{x_1}{|x_1|} = \dfrac{[1,-1,-1,1]^T}{2} = [0.5,-0.5,-0.5,0.5]^T$

   $q_2 = \dfrac{x_2 - (q_1^T x_2)q_1}{|x_2 - (q_1^T x_2)q_1|} = \dfrac{[2,1,0,1]^T - (1-0.5+0.5)[0.5,-0.5,-0.5,0.5]^T}{\dots} =$
   $\dfrac{[1.5,1.5,0.5,0.5]^T}{\sqrt{5}} = \left[\dfrac{3}{2\sqrt{5}}, \dfrac{3}{2\sqrt{5}}, \dfrac{1}{2\sqrt{5}}, \dfrac{1}{2\sqrt{5}}\right]$

   $q_3 = \dfrac{x_3 - (q_2^T x_3)q_2 - (q_1^T x_3)q_1}{|\dots|} =$
   $\dfrac{[2,2,1,2]^T - (\frac{6+6+1+2}{2\sqrt{5}})[\frac{3}{2\sqrt{5}}, \frac{3}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}, \frac{1}{2\sqrt{5}}] - (1-1-0.5+1)[0.5,-0.5,-0.5,0.5]^T)}{\dots} =$
   $\dfrac{[2,2,1,2]^T - [\frac{45}{20}, \frac{45}{20}, \frac{15}{20}, \frac{15}{20}]^T - [0.25,-0.25,-0.25,0.25]^T}{\dots} = \dfrac{[-0.5,0,0.5,1]^T}{\frac{\sqrt{6}}{2}} =$
   $\left[-\dfrac{1}{\sqrt{6}}, 0, \dfrac{1}{\sqrt{6}}, \sqrt{\dfrac{2}{3}}\right]$

8. $QR = \begin{bmatrix} 0.5 & \frac{1.5}{\sqrt{5}} & -\frac{1}{\sqrt{6}} \\ -0.5 & \frac{1.5}{\sqrt{5}} & 0 \\ -0.5 & \frac{0.5}{\sqrt{5}} & \frac{1}{\sqrt{6}} \\ 0.5 & \frac{0.5}{\sqrt{5}} & \sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0.5 \\ 0 & \sqrt{5} & 1.5\sqrt{5} \\ 0 & 0 & \sqrt{\frac{3}{2}} \end{bmatrix}$. $R^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{5}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{5}} & -\sqrt{\frac{3}{2}} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{bmatrix}$.

   $x = R^{-1}Q^Tb = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{5}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{5}} & -\sqrt{\frac{3}{2}} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ \frac{1.5}{\sqrt{5}} & \frac{1.5}{\sqrt{5}} & \frac{0.5}{\sqrt{5}} & \frac{0.5}{\sqrt{5}} \\ -\frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ -2 \\ 0 \end{bmatrix} =$
   $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2\sqrt{5}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{5}} & -\sqrt{\frac{3}{2}} \\ 0 & 0 & \sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} 3.5 \\ -\frac{\sqrt{5}}{2} \\ -2\sqrt{\frac{2}{3}} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 1.5 \\ -\frac{4}{3} \end{bmatrix}$

9. $v_1 = [1,2,3]^T$.

   $v_2 = [1,0,0]^T - \dfrac{1}{14}[1,2,3]^T = [\dfrac{13}{14}, -\dfrac{2}{14}, -\dfrac{3}{14}]^T$.

   $v_3 = [0,1,0]^T - \dfrac{2}{14}[1,2,3]^T - (-\dfrac{2}{14})\dfrac{1}{\frac{13}{14}}[\dfrac{13}{14}, -\dfrac{2}{14}, -\dfrac{3}{14}]^T = [0,1,0]^T - \dfrac{1}{7}[1,2,3]^T +$
   $[\frac{1}{7}, -\frac{4}{91}, -\frac{3}{91}]^T = [0, \frac{63}{91}, -\frac{42}{91}]^T$

10. Characteristic equation is always $\det(X - \lambda I) = 0$.

(a) $(1-\lambda)(3-\lambda)(1-\lambda) - 6 + 2(3-\lambda) - 3(1-\lambda) = -\lambda^3 + 5\lambda^2 - 7\lambda + 3 - 6 + 6 - 2\lambda - 3 + 3\lambda = -\lambda^3 + 5\lambda^2 - 6\lambda = -\lambda(\lambda^2 - 5\lambda + 6) = -\lambda(\lambda-3)(\lambda-2)$.

So the eigenvalues are $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 3$.

$$\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 3 & 3 & 0 \\ -2 & 1 & 1 & 0 \end{array} \implies v_1 = [0, 1, -1]^T.$$

$$\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ -2 & 1 & -1 & 0 \end{array} \implies v_2 = [2, 3, -1]^T.$$

$$\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ -2 & 1 & -2 & 0 \end{array} \implies v_3 = [-1, 2, 0]^T.$$

Algebraic and geometric multiplicities of all eigenvalues are 1 and the matrix is not defective.

(b) $(1-\lambda)^3 = 0$, so the only eigenvalue is $\lambda = 1$

$$\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \implies v_1 = [1, 0, 0]^T$$

Algebraic multiplicity is 3, geometric multiplicity is 1 so the matrix is defective.

(c) $(1-\lambda)^3 = 0$, so the only eigenvalue is $\lambda = 1$

$$\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \implies v_1 = [1, 0, 0]^T, v_2 = [0, 0, 1]^T$$

Algebraic multiplicity is 3, geometric multiplicity is 2 so the matrix is defective.

(d) D is a diagonal matrix, so the only eigenvalue is 1 (with multiplicity 3) and the eigenvectors are e_1, e_2, e_3.

(e) $$\begin{vmatrix} -4-\lambda & 1 & 1 & 1 \\ -16 & 3-\lambda & 4 & 4 \\ -7 & 2 & 2-\lambda & 1 \\ -11 & 1 & 3 & 4-\lambda \end{vmatrix} = \begin{vmatrix} -4-\lambda & 1 & 0 & 0 \\ -16 & 3-\lambda & \lambda+1 & 0 \\ -7 & 2 & -1 & \lambda-1 \\ -11 & 1 & 3-\lambda & 1-\lambda \end{vmatrix} =$$

$$\begin{vmatrix} -4-\lambda & 1 & 0 & 0 \\ -16 & 3-\lambda & \lambda+1 & 0 \\ -7 & 2 & -1 & \lambda-1 \\ -18 & 3 & 2-\lambda & 0 \end{vmatrix} = (\lambda-1) \begin{vmatrix} -4-\lambda & 1 & 0 \\ -16 & 3-\lambda & \lambda+1 \\ -18 & 3 & 2-\lambda \end{vmatrix} = (\lambda-1)((-4-\lambda)(3-\lambda)(2-\lambda) - 18(\lambda+1) + 3(\lambda+1)(\lambda+4) + 16(2-\lambda)) = (1-\lambda)^3(\lambda-2)$$

11. Let $Av = \lambda v$, where λ is an eigenvalue and v is an eigenvector.

(a) We will use induction. Base case: $Av = \lambda v$.

Suppose $A^{k-1}v = \lambda^{k-1}v$. Then $A^k v = A\lambda^{k-1}v = \lambda^{k-1}Av = \lambda^{k-1}\lambda v = \lambda^k v$.

So λ^k is an eigenvalue of A^k for any $k = 2, 3, \ldots$.

(b) $Av = \lambda v \implies A^{-1}Av = A^{-1}\lambda v \implies \frac{1}{\lambda}v = A^{-1}v$ so $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

(c) λ is an eigenvalue of $A \iff \det(A - \lambda I) = 0$. We have $\det(A + \alpha I - (\lambda + \alpha)I) = \det(A - \lambda I) = 0$, so $\lambda + \alpha$ is an eigenvalue of $A + \alpha I$.

12. $(4-\lambda)(2-\lambda)-3=0 \implies \lambda^2-6\lambda+5=0 \implies \lambda \in \{1,5\}$.

    The eigenvector corresponding to $\lambda = 1$ is $v_1 = [1,-1]^T$.

    For $\lambda = 5$, we have $(A-5I)v_2 = 0$.

    $$\begin{array}{cc|c} -1 & 3 & 0 \\ 1 & -3 & 0 \end{array}$$

    So $v_2 = [3,1]^T$.

    Thus we have $S = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ and $A = S^{-1}\Lambda S$.

    Finally $A^{100} = S^{-1}\Lambda^{100}S = \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5^{100} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & -\frac{3 \cdot 5^{100}}{4} \\ \frac{1}{4} & \frac{5^{100}}{4} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix} =$
    $\begin{bmatrix} \frac{3 \cdot 5^{100}+1}{4} & \frac{3 \cdot (5^{100}+1)}{4} \\ -\frac{5^{100}-1}{4} & -\frac{5^{100}-3}{4} \end{bmatrix}$.

13. (a) λ is an eigenvalue of A, so $\det(A-\lambda I) = 0$.

        We have $\det(q(A)-q(\lambda)I) = \det((A^3-\lambda^3 I)-2(A^2-\lambda^2 I)-(A-\lambda I)) = \det((A-\lambda I)B) = \det(A-\lambda I)\det(B) = 0\det(B) = 0$. So $q(\lambda)$ is an eigenvalue of $q(A)$.

    (b) Eigenvalues of $q(A)$ are $2, 0, 8$.

        Eigenvalues of $q(B), q(C), q(D)$ are 0.

        The only eigenvalue of $q(E)$ is 0.

14. A has distinct eigenvalues, so their eigenvectors form a basis. For every eigenvector v wih eigenvalue λ, we have $p(A)v = p(\lambda)v = 0v = 0$.

    We now have two matrices, $p(A)$ and 0, that give the same result (namely 0) when applied to the elements of a basis (the eigenvectors of A). Therefore they are equal.

15. Let $\lambda = \lambda_r + i\lambda_i$ and $x = x_r + ix_i$.

    Then we have $Ax_r + iAx_i = \lambda_r x_r - \lambda_i x_i + i(\lambda_r x_i + \lambda_i x_r)$. Therefore $Ax_r = \lambda_r x_r - \lambda_i x_i$ and $Ax_i = \lambda_r x_i + \lambda_i x_r$.

    Thus $A\bar{x} = Ax_r - iAx_i = \lambda_r x_r - \lambda_i x_i - \lambda_r x_i - \lambda_i x_r = (\lambda_r - i\lambda_i)(x_r - ix_i) = \bar{\lambda}\bar{x}$.

16. We have $\bar{x}^T(Ax) = \lambda\bar{x}^T x \implies (\bar{x}^T(Ax))^T = \lambda(\bar{x}^T x)^T \implies (Ax)^T\bar{x} = \lambda x^T\bar{x} \implies x^T A\bar{x} = \lambda x^T\bar{x}$. But we know $\bar{\lambda}$ is also an eigenvalue of A, with corresponding eigenvector $\bar{x}$ so $x^T(A\bar{x}) = \bar{\lambda}x^T\bar{x}$. Therefore $\lambda = \bar{\lambda}$, so $\lambda \in \mathbb{R}$.

17. As $\det(S^{-1})\det(S) = 1$, we have $\det(A-\lambda I) = \det(S^{-1})\det(A-\lambda I)\det(S) = \det(S^{-1}(A-\lambda I)S) = \det(S^{-1}AS - \lambda S^{-1}IS) = \det(S^{-1}AS - \lambda I)$. Thus, A and $S^{-1}AS$ have the same characteristic polynomial and therefore same eigenvalues.

18. If A is diagonalizable, then it can be written as $S^{-1}\Lambda S$, so S is invertible, so $u_1, u_2, \ldots, u_n$ are linearly independent.

    If $u_1, u_2, \ldots, u_n$ are linearly independent then S^{-1} exists and we have

    $AS = [\lambda_1 u_1, \lambda_2 u_2, \ldots, \lambda_n u_n] = [u_1, u_2, \ldots, u_n]\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) = S\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \implies S^{-1}AS = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, so A is diagonalizable.

Applications

1. $u \times v = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 0 & -2 & 1 \end{vmatrix} = (2-2)i - j - 2k = [0,-1,-2]$.

2. The volume of a tetrahedron is one sixth of the volume of the associated parallelepiped, which is $u \cdot (v \times w) = \begin{vmatrix} u \\ v \\ w \end{vmatrix}$.

   For the given u, v, w we have $V = \frac{1}{6}\begin{vmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \frac{4+2}{6} = 1$.

3. (a) $\begin{vmatrix} a & b & 1 \\ 1 & 2 & 1 \\ 2 & 5 & 1 \end{vmatrix} = 0 \implies 2a + 2b + 5 - 4 - 5a - b = 0 \implies b + 1 = 3a$

   (b) $\begin{vmatrix} a & b & 1 \\ 2 & 4 & 1 \\ -1 & 6 & 1 \end{vmatrix} = 0 \implies 4a + 12 - b + 4 - 6a - 2b = 0 \implies 2a + 3b = 16$

4. Matrix of the system is $A = \begin{bmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{bmatrix}$.

   Suppose one of the rows can be written as a linear combination of the others. Thus, we have $[1, a, a^2] = k[1, b, b^2] + (1-k)[1, c, c^2]$. As the vectors are different, $k \notin \{0, 1\}$. Then $a = kb + (1-k)c \wedge a^2 = kb^2 + (1-k)c^2 \implies (kb + (1-k)c)^2 = kb^2 + (1-k)c^2 \implies k(k-1)b^2 + (1-k)(1-k-1)c^2 = 2k(k-1)bc \implies b^2 + c^2 = 2bc \implies b = c$, which is false.

   Thus the rows are linearly independent, which means that the system has a unique solution.

5. Let $P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$. We have $x_1 = [1,1,1,1]^T$ and $x_2 = [0,1,2,3]^T$.

   Then $q_1 = [0.5, 0.5, 0.5, 0.5]^T$ and $q_2 = \dfrac{x_2 - (q_1^T x_2)q_1}{\cdots} = \dfrac{[0,1,2,3]^T - [1.5, 1.5, 1.5, 1.5]^T}{\cdots} = \dfrac{[-1.5, -0.5, 0.5, 1.5]}{\sqrt{5}}$.

   Finally $P = \begin{bmatrix} 0.5 & -0.3\sqrt{5} \\ 0.5 & -0.1\sqrt{5} \\ 0.5 & 0.1\sqrt{5} \\ 0.5 & 0.3\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & \sqrt{5} \end{bmatrix} = QR$.

   We have to solve $P^T P x = P^T b$, and the solution is $x = R^{-1}Q^T b = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} \end{bmatrix}$

   $$\begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ -0.3\sqrt{5} & -0.1\sqrt{5} & 0.1\sqrt{5} & 0.3\sqrt{5} \end{bmatrix} \begin{bmatrix} 30 \\ 40 \\ 60 \\ 100 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 115 \\ 23\sqrt{5} \end{bmatrix} = \begin{bmatrix} 23 \\ 23 \end{bmatrix}.$$

   So $p(x) = 23 + 23t$ is the least squares fit. The expected population in 2020 is 483.

6. We know that the largest eigenvalue is positive.

   $P^T[1,1,1]^T = [1,1,1]^T$, so its eigenvalue is 1.

   Let $x = 10\lambda$ where λ is an eigenvalue of P. We have $(5-x)(2-x)(1-x)+4*3*3+6*2*4-3*6*(2-x)-4*3*(5-x)-4*2*(1-x) = -x^3+8x^2+21x-10 = (x-10)(-x^2-2x+1) = (x-10)(x+1-\sqrt{2})(x+1+\sqrt{2})$, so the eigenvalues are 1, $\frac{-1-\sqrt{2}}{10}$, $\frac{-1+\sqrt{2}}{10}$.

   Steady state vector is $[\frac{15}{8}, \frac{27}{32}, 1]^T$.

7. $\begin{vmatrix} -1.5 & 4 & 3 \\ 0.5 & -1.5 & 0 \\ 0 & 0.25 & -1.5 \end{vmatrix} = (-1.5)^3 + 0.5 \cdot 0.25 \cdot 3 + 0.5 \cdot 1.5 \cdot 4 = 0$, so 1.5 is an eigenvalue. If we write $L = S^{-1}\Lambda S$, we have $L^n = S^{-1}\Lambda^n S$. As 1.5 is one of the numbers on the diagonal of Λ, Λ^n will never be 0.

   We will use the following PDL script that implements the power method:

```
#!/usr/bin/perl
use PDL;
$a = pdl [[0,1,3/4],[1/2,0,0],[0,1/4,0]];
$y = pdl ([1,1,1])->transpose;

while (max abs ($y - $x) > 0.1) {
    $x = $a x $y;
    $x /= max $x;
    ($l) = list ($x->transpose x $a x $x / ($x->transpose x $x));
    printf "Step #%02d: [%.2f, %.2f, %.2f] L=%.2f\n", ++$s, list ($x),  $l;
    ($x, $y) = ($y, $x);
}
```

   The script prints the following output:

```
Step #01: [1.00, 0.29, 0.14] L=0.50
Step #02: [0.79, 1.00, 0.14] L=0.79
Step #03: [1.00, 0.35, 0.23] L=0.61
Step #04: [1.00, 0.95, 0.17] L=0.82
Step #05: [1.00, 0.46, 0.22] L=0.70
Step #06: [1.00, 0.80, 0.18] L=0.82
Step #07: [1.00, 0.54, 0.21] L=0.74
Step #08: [1.00, 0.72, 0.19] L=0.81
Step #09: [1.00, 0.58, 0.21] L=0.77
Step #10: [1.00, 0.68, 0.20] L=0.80
Step #11: [1.00, 0.60, 0.21] L=0.78
```

   Thus, the largest eigenvalue is 0.78, so all eigenvalues are smaller than 1. Using the same logic as before, we have $\Lambda^n \to 0$ when $n \to \infty$, so the population will not survive.

8. $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

We are looking for the eigenvalue that corresponds to the largest eigenvalue of A.

Using the following PDL script that computes the ranking vector using the power method,

```
#!/usr/bin/perl
use PDL;
$a = pdl [[0,1,1,0],[0,0,0,1],[0,1,0,1],[1,0,0,0]];
$y = pdl ([1,1,1,1])->transpose;

while (max abs ($y - $x) > 0.05) {
$x = $a x $y;
$x /= max $x;
printf "Step #%02d: [%.3f, %.3f, %.3f, %.3f]\n", ++$s, list ($x);
($x, $y) = ($y, $x);
}

printf "\nScaled: [%.2f, %.2f, %.2f, %.2f]\n", list $y / sum $y
```

we get the following output:

```
Step #01: [1.000, 0.500, 1.000, 0.500]
Step #02: [1.000, 0.333, 0.667, 0.667]
Step #03: [1.000, 0.667, 1.000, 1.000]
Step #04: [1.000, 0.600, 1.000, 0.600]
Step #05: [1.000, 0.375, 0.750, 0.625]
Step #06: [1.000, 0.556, 0.889, 0.889]
Step #07: [1.000, 0.615, 1.000, 0.692]
Step #08: [1.000, 0.429, 0.810, 0.619]
Step #09: [1.000, 0.500, 0.846, 0.808]
Step #10: [1.000, 0.600, 0.971, 0.743]
Step #11: [1.000, 0.473, 0.855, 0.636]
Step #12: [1.000, 0.479, 0.836, 0.753]
Step #13: [1.000, 0.573, 0.938, 0.760]
Step #14: [1.000, 0.503, 0.883, 0.662]
Step #15: [1.000, 0.478, 0.841, 0.721]
Step #16: [1.000, 0.547, 0.909, 0.758]
Step #17: [1.000, 0.521, 0.896, 0.687]
Step #18: [1.000, 0.484, 0.852, 0.706]

Scaled: [0.33, 0.16, 0.28, 0.23]
```

Thus the order is A, C, D, B.

9. We are looking for the eigenvalue that corresponds to the largest eigenvalue of S^T.

   Using a PDL script similar to the one above we get:

```
Step #01: [0.667, 1.000, 0.333, 0.333]
Step #02: [0.500, 1.000, 0.500, 0.750]
Step #03: [0.714, 1.000, 0.286, 0.571]
Step #04: [0.545, 1.000, 0.455, 0.636]
Step #05: [0.667, 1.000, 0.333, 0.611]
Step #06: [0.586, 1.000, 0.414, 0.621]
Step #07: [0.638, 1.000, 0.362, 0.617]
Step #08: [0.605, 1.000, 0.395, 0.618]
Step #09: [0.626, 1.000, 0.374, 0.618]
Step #10: [0.613, 1.000, 0.387, 0.618]
Step #11: [0.621, 1.000, 0.379, 0.618]
Step #13: [0.619, 1.000, 0.381, 0.618]
Step #14: [0.617, 1.000, 0.383, 0.618]
Step #15: [0.618, 1.000, 0.382, 0.618]
Step #16: [0.618, 1.000, 0.382, 0.618]

Scaled: [0.236, 0.382, 0.146, 0.236]
```

   Thus the order is 2, 1, 4, 3 (or 2, 4, 1, 3 since 4 and 1 have the same ranking).