Logic and Proof (Sheet #3)

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1. $\forall x \exists y f(y) = x$
   $\forall a \forall b \forall c R(a,a) \land (R(a,b) \leftrightarrow R(b,a)) \land (R(a,b) \land R(b,c) \implies R(a,c))$
   $\exists a \exists b \exists c \forall d P(a) \land P(b) \land P(c) \land \neg P(d)$

2. $\exists x R(x,0)$ is satisfied by $\mathcal{B}$ but not by $\mathcal{A}$.

   $\exists x R(0,x) \land R(x,1)$ is satisfied by $\mathcal{C}$ but not by $\mathcal{B}$.

3. Rectified: $\forall z \exists y (Q(x, g(y), z) \lor \neg \forall t P(t)) \land \neg \forall a \exists b \neg R(f(b,a), a)$

   Prenex: $\forall z \exists y ((Q(x, g(y), z) \lor \exists t \neg P(t)) \land \exists a \forall b R(f(b,a), a))$
   $\forall z \exists y \exists a \forall b \exists t ((Q(x, g(y), z) \lor P(t)) \land R(f(b,a), a))$.

   Skolem: $\forall z \forall b ((Q(x, g(y(z)), z) \lor P(t(z,b))) \land R(f(b, a(z)), a(z)))$,

4. (a) Incorrect. ???
   (b) Correct. If $\exists x$ such that $\neg P(x)$, then that x satisfies $P(x) \to \forall y P(y)$. Otherwise we have $\forall x P(x)$ so $P(x) \to \forall y P(y)$ is valid. In both cases $\exists x (P(x) \to \forall y P(y))$ is valid.
   (c) Correct, since the newly added constant term can take any value. So if $F[c/x]$ is valid, we know F is valid for any value of x.

5. (a) Inductive definition. Base case: $F = P_1$ so $F[...] = G_1$.
   $F = \neg G \implies F[...] = \neg G[...]$
   $F = G \land H \implies F[...] = G[...] \land H[...]$.
   (b) Use induction. Base case: $F = P_1$ then $F' = P_1$ so $F[...] = G_1 = G'_1 = F'[...']$.
   If $F = \neg G$ and $F' = \neg G'$ we have $F[...] = (\neg G)[...] = \neg G[...]$ which by induction hypothesis is $\neg G'[...'] = (\neg G')[...'] = F'[...']$.
   If $F = G \land H$ and $F' = G' \land H'$ then $F[...] = (G \land H)[...] = G[...] \land H[...]$ which by induction hypothesis is $G'[...'] \land H'[...'] = (G' \land H')[...'] = F'[...']$.
   (c) No functions, finite number of constants, variables and predicates means finite number of terms (say T terms). Each of these terms can be considered a propositional variable, and with only T variables we only have 2^{2^T} possible logically distinct propositional formulas.
   (d)

6. We will use induction. Base case: F is atomic. This is true due to (P1).

   Induction step: Assume $\forall A, B. A \sim B$, $A \models F_1$ iff $B \models F_1$ and $A \models F_2$ iff $B \models F_2$.

   Take $F = \neg F_1$. Then $A \models F \iff A \not\models F_1 \iff B \not\models F_1 \iff B \models F$.

   Take $F = F_1 \land F_2$. Then $A \models F \iff A \models F_1 \land A \models F_2 \iff B \models F_1 \land B \models F_2 \iff B \models F$.

   Take $F = \exists x F_1$. Then $A \models F$ iff $\exists a \in U_a$ such that $A_{x \to a} \models F_1$. Then by (P2) $\exists b \in U_b$ such that $A_{x \to a} \sim B_{x \to b}$, and by the induction hypothesis we get $B_{x \to b} \models F_1$ wchich is equivalent to $B \models F$. So $A \models F$ implies $B \models F$, and we can similarly show $B \models F$ implies $A \models F$. So $A \models F$ iff $B \models F$.

Thus if our property is true for F_1 and F_2, it is true for $\neg F_1, F_1 \wedge F_2, \exists x F_1$, and it is true for any atomic formula. Since all formulas are composed from atomic formulas, negation, conjunction and existential quantifiers, we can conclude that the property is true for all formulas.

7. (a) Example: $\exists x_1 \exists x_2 \ldots \exists x_n (\bigwedge_{i<j} (x_i R x_j \wedge \neg x_j R x_i))$.

   (b)