Probabilistic Model Checking (Sheet #1)

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1. (a) $\frac{1}{3} + \frac{2}{3}(\frac{1}{4} + \frac{1}{4}) = \frac{2}{3}$

   (b) s_3 can only be reached from s_0, s_5, s_4, s_3. The probability is 1 from s_3 and s_4, $\frac{1}{2}\frac{1}{4} = \frac{1}{8}$ from s_0, and $\frac{1}{4} + \frac{1}{2}\frac{2}{3}\frac{1}{4} = \frac{1}{3}$ from s_5 (because we can go $s_5 s_4 s_3$, or $s_5 s_0 s_5 s_4 s_3$ and both options take no more than 4 transitions).

   (c) Same as before, but s_0 and s_5 get more complicated.

   s_0 has probability $(\frac{2}{3}\frac{1}{4})(1 + \frac{1}{3} + \frac{1}{9} + \ldots)$. The first term represents the probability of going $s_0 s_5 s_4 s_3$, and the second term represents the probability of doing $s_0 s_5 s_0$ 0 times, 1 time, 2 times, and so on. The total probabiliy is $\frac{1}{6}\frac{3}{2}$ which is $\frac{1}{4}$.

   Now we can use this to compute s_5's probability, which is $\frac{1}{4} + \frac{1}{2}\frac{1}{4}$ (because we go right with probability $\frac{1}{4}$, and we go to s_0 with probability $\frac{1}{2}$), so this is $\frac{3}{8}$.

   (d) To not reach the states $A \cup B$ means to get stuck in state s_1 or s_6 or in the $s_0 s_5$ loop. All of these paths have probability 0, so the probability of reaching a state in $A \cup B$ is 1.

   (e) With the same logic as above, the probability of getting stuck in one of those loops is 0, so the probability of visiting $A \cup B$ infinitely often is also 1.

2. (a) We can represent this set as the complement of the set of all paths that contain a state not in A. This set is the union of all cylinder sets $\text{Cyl}(\omega)$ where ω contains only states in A, except for the last state. There are a countable number of such cylinder sets, so their union is measurable, and its complement is also measurable.

   (b) We can represent this set as the intersection of the set from part (a) and the set of paths whose fifth state is in B. This latter set is the union of cylinder sets $\text{Cyl}(\omega)$ where ω has length 5 (or 6, depending on how we interpret the problem statement) and its last state is in B. There is a finite number of such cylinder sets, so their union is measurable. The intersection is measurable too, and that is the set we are looking for.

3. The states are a_i with $0 \le i \le n$, representing that player 1 has i coins and player 2 has $n - i$ coins. Starting state is a_m. We have transitions with probability 0.5 from a_i to a_{i-1} and to a_{i+1} for all $0 < i < n$. We have transitions from a_0 to itself with probability 1, and from a_n to itself with probability n. We label state a_0 with "win2" and state a_n with "win1".

   With probability 1, we will reach one of the two BSCCs (they are $\{a_0\}$ and $\{a_n\}$). Both of these represent termination of the game, so the game terminates with probability 1.

   Player 1 has a better chance of winning than player 2 if the probability of eventually reaching "win1" is more than one half. This can be written as $\mathrm{P}_{>0.5}[\mathrm{F}\,\text{win1}]$.

   The last statement can be expressed as $\mathrm{P}_{\le 0.1}[\mathrm{F}^{\le 5}\,\mathrm{P}_{>0}[\mathrm{X}\,\text{win1}]]$.

4. $\mathrm{E\,G}\,a$ means there exists a path starting from here such that all its states satisfy a. $\mathrm{P}_{>0}[\mathrm{G}\,a]$ means that it is possible to find a path starting from here such that all its states satisfy a. These two statements are equivalent.

   We know that $s \models \mathrm{P}_{\geq 1}[\mathrm{G}\,a] \iff s \models \mathrm{P}_{\leq 0}[\mathrm{F}\,\neg a] \iff s \not\models \mathrm{P}_{>0}[\mathrm{F}\,\neg a] \iff s \not\models \mathrm{E\,F}\,\neg a \iff s \models AGa$. So the first pair of statements are also equivalent.

5. $P = \begin{bmatrix} 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & p & 1-p \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$

   $\pi(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   We can compute the long-run probability of being in s_2 to be 0.25.

   For state s_4, the probability of reaching its BSCC is $\frac{1}{2}$. So the probability of being in $\{s_2, s_4\}$ is $0.25 + 0.5 * p$ where p is the probability of being in s_4 given that we are in its BSCC. We need to solve:

   $x \cdot P = x$ with x starting as $\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$.