Probabilistic Model Checking (Sheet #4)

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1. Parts (a) and (b) are on a separate sheet. For part (c), let x_{ij} be the probability to reach s_3q_3 from s_iq_j. We can see $x_{33} = 1, x_{22} = x_{02} = 0$. Then $x_{11} = 0$ and $x_{21} = \frac{1}{2} \cdot (1 + 0) = 0.5$ and $x_{30} = \frac{1}{2}(x_{30} + x_{21}) = 0.5$ and $x_{00} = \frac{1}{4}(x_{00} + x_{30} + x_{21} + x_{11}) = 0.75$.

   So the probability of reaching s_3q_3 from the initial state is 0.75. This is the probability of NOT satisfying the safety property. Therefore the probability of satisfying the safety property from the initial state is 0.25.

2. First we construct a NBA for the LTL property and notice it is actually a DBA, so we can use it directly. Now we make a product DTMC between the given DTMC and this DBA. See separate sheet for pictures of NBA and product DTMC.

   The product DTMC has a single BSCC, $\{s_2q_0, s_3q_0\}$. By the fundamental property we will reach it and visit all its states infinitely often with probability 1. Since this BSCC contains accepting states (in fact both of its states are accepting), we conclude that all states of the original DTMC satisfy the LTL property $\mathrm{G}(a \to \mathrm{F}\, c)$ with probability 1.

3. An LTL formula for this automaton is $(a \vee \mathrm{X}\, b) \wedge a\, \mathrm{U}(b \vee \mathrm{X}\, b)$. The left branch of the conjunction rules out the path $q_0q_2q_4$. If $\pi \not\models b \vee \mathrm{X}\, b$, then the second branch covers all paths starting q_0q_1. If $\pi \models b \vee \mathrm{X}\, b$, there are two cases:

   Case 1: $\pi \models \mathrm{X}\, b$, in which case we have either $q_0q_1q_3^\omega$ or $q_0q_2q_3^\omega$.

   Case 2: $\pi \models b$ (but $\pi \not\models \mathrm{X}\, b$), in which case the left branch tells us $\pi \models a$ so we have $q_0q_1q_3^\omega$.

   An MDP with minimum probability 0, maximum probability 1 is drawn on a separate sheet. Choosing one action produces traces like ab^ω which satisfy the property, choosing the other action produces traces like a^ω which do not satisfy the property.

4. (a) To reach s_4, we want to choose action d in s_3. Now we reach s_4 with probability 1 in s_3, s_4, and 0 in s_2. From s_0 we want to choose action b which gets us to s_4 with probability 1.
To reach s_2, we note $S^{\text{yes}} = \{s_2\}$ and $S^{\text{no}} = \{s_4\}$. Now we want to minimize $\sum_{s \in S^?} x_s$ with constraints:

$$
\begin{aligned}
x_1 &\geq \frac{1}{2}x_3 + \frac{1}{2} \\
x_3 &\geq \frac{1}{3}x_1 && \text{(action c)} \\
x_3 &\geq 0 && \text{(action d)} \\
x_0 &\geq x_3 && \text{(action b)} \\
x_0 &\geq \frac{1}{2}x_1 && \text{(action a)}
\end{aligned}
$$

from which we get $x_3 = \frac{1}{3}x_1$, $\frac{5}{6}x_1 = \frac{1}{2}$ so $x_1 = \frac{3}{5}$ and $x_3 = \frac{1}{5}$ and so $x_0 = \frac{3}{10}$.
The optimal strategy is to choose actions a and c. We can see this by looking at the values we got and the constraints, seeing which constraints are actually equal (instead of $>$), and choosing the actions associated with them.

(b) We minimize the same sum as above, with constraints:

$$
\begin{aligned}
x_1 &\geq \frac{1}{2}x_3 + \frac{1}{2} \\
x_3 &\geq qx_1 && \text{(action c)} \\
x_3 &\geq 0 && \text{(action d)} \\
x_0 &\geq x_3 && \text{(action b)} \\
x_0 &\geq px_1 && \text{(action a)}
\end{aligned}
$$

We always choose c instead of d, so $x_3 = qx_1$. We choose a instead of b when $px_1 > x_3 = qx_1$. So we choose a instead of b when $p > q$ and b instead of a when $q > p$. When $p = q$ choosing either a or b gives us the same probability of reaching s_2.