Probability (Sheet #1)

Marius Gavrilescu

1. (a) $6!$
   (b) $5!$
   (c) $2 \cdot 3! \cdot 3!$
   (d) $6! - 5!$
2. (a) $\binom{7}{2\ 2\ 2\ 1}$
   (b) $\frac{\binom{8}{3}}{2^8}$
   (c) $\frac{\binom{9}{3\ 2\ 2\ 1\ 1}}{6^9}$
3. If a $(r+1)$-subset of $[n+1]$ has $(k+1)$ as its largest element, then its other r elements are distinct integers between 1 and k. Thus, there are $\binom{k}{r}$ such subsets.

   There are $\binom{n+1}{r+1}$ $(r+1)$-subsets of $[n+1]$. For every such subset, its largest element is at least $r+1$ and at most $n+1$. Let $k \in \{r, r+1, \ldots, n\}$. Then, there are $\binom{k}{r}$ $(r+1)$-subsets of $[n+1]$ with $(k+1)$ as their largest element. Summing over the possible values of k we get:

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$

4. (a) $\mathbb{P}(\Omega) = \mathbb{P}(\Omega \cup \varnothing) = \mathbb{P}(\Omega) + \mathbb{P}(\varnothing) \implies \mathbb{P}(\varnothing) = 0$
   (b) Let $C = A \setminus B$ and $D = A \cap B$. Then $A = C \cup D$.
   $\mathbb{P}(A) = \mathbb{P}(C \cup D) = \mathbb{P}(C) + \mathbb{P}(D) \implies \mathbb{P}(C) = \mathbb{P}(A) - \mathbb{P}(D)$.
   (c) Let $C = A \cap B$, $D = A \setminus B$ and $E = B \setminus A$. Obviously, $A \cup B = C \cup D \cup E$.
   $\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = \mathbb{P}(C \cup D) + \mathbb{P}(C \cup E) - \mathbb{P}(C) = \mathbb{P}(C) + \mathbb{P}(D) + \mathbb{P}(C) + \mathbb{P}(E) - \mathbb{P}(C) = \mathbb{P}(C) + \mathbb{P}(D) + \mathbb{P}(E) = \mathbb{P}(C \cup D \cup E) = \mathbb{P}(A \cup B)$
5. (a) $A^c \cap (B \cup C)$
   (b) $\mathbb{P}(A^c \cap (B \cup C)) = \mathbb{P}((B \cup C) \setminus A) = \mathbb{P}((A \cup B \cup C) \setminus A) = \mathbb{P}(A \cup B \cup C) - \mathbb{P}((A \cup B \cup C) \cap A) = \mathbb{P}(A \cup B \cup C) - \mathbb{P}(A)$.
   Apply inclusion-exclusion formula on $\mathbb{P}(A \cup B \cup C)$ and get
   $\mathbb{P}(A \cup B \cup C) - \mathbb{P}(A) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C) - \mathbb{P}(A \cap C) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B \cap C) - \mathbb{P}(A) = \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C) - \mathbb{P}(A \cap C) - \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B \cap C)$
6. (a) There are $n!\binom{365}{n}$ cases where no two people have the same birthday, and 365^n cases in total. Thus, the probability that at least two people have the same birthday is:

$$p(n) = \frac{365^n - n!\binom{365}{n}}{365^n}$$

   $p(22) = 0.48$ and $p(23) = 0.51$ so for $n = 23$ the probability is larger than $\frac{1}{2}$.
   (b) The probability of one person not having the same birthday as you is $\frac{364}{365}$. The probablity of no person having the same birthday as you is $(\frac{364}{365})^n$.
   Thus, the probability that at least one person has the same birthday as you is:

$$p'(n) = 1 - \left(\frac{364}{365}\right)^n$$

   $p'(252) = 0.499$ and $p'(253) = 0.5$ so for $n = 253$ the probability is larger than $\frac{1}{2}$.
7. (a) If the first key is fixed (first two keys are fixed), there are $(n-1)!$ ways to arrange the other $n-1$ keys ($(n-2)!$ ways to arrange the other $n-2$ keys). In both cases, the total number of arrangements is $n!$.

(b) $\binom{n}{2}$

(c) $\mathbb{P}(\cup_{1=1}^{n} A_i) = \frac{1}{n!}\left(\binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \ldots + (-1)^{n+1}\binom{n}{n}(n-n)!\right) =$

$= \frac{1}{n!}\sum_{k=1}^{n}(-1)^{k+1}\frac{n!}{(n-k)!k!}(n-k)! = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!}$

(d) $p_n(0) = 1 - \mathbb{P}(\cup_{i=1}^{n} A_i) = 1 - \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k!} = \sum_{k=0}^{n}\frac{(-1)^k}{k!}$

If exactly r keys are on the correct hook, we can ignore them and consider that we have $n-r$ hooks and $n-r$ keys, none of which are on the correct hook.

The probability of the r keys being on the right position is $\frac{}{r!\binom{n}{r}}$

Since the probability of the r keys being on the right position is $\frac{(n-r)!}{n!}$, we get $p_n(r) = \frac{1}{r!}p_{n-r}(0) = \frac{1}{r!}\sum_{k=0}^{n-r}\frac{(-1)^k}{k!}$.

(e) The base case is 4(c), proved above.

Suppose $\mathbb{P}(\cup_1^n A_i) = \sum_{i=1}^{n}\mathbb{P}(A_i) - \sum_{1\leq i<j\leq n}\mathbb{P}(A_i \cap A_j) + \ldots + (-1)^{n+1}\mathbb{P}(\cap_{1\leq i\leq n} A_i)$.

Then:

$\mathbb{P}\left(\cup_1^{n+1} A_i\right) =$

$\mathbb{P}\left((\cup_1^n A_i) \cup A_{n+1}\right) =$

$\mathbb{P}(\cup_1^n A_i) + \mathbb{P}(A_n) - \mathbb{P}((\cup_1^n A_i) \cap A_n) =$

$\mathbb{P}(\cup_1^n A_i) + \mathbb{P}(A_n) - \mathbb{P}(\cup_1^n (A_i \cap A_n)) =$

$\sum_{i=1}^{n}\mathbb{P}(A_i) - \sum_{1\leq i<j\leq n}\mathbb{P}(A_i \cap A_j) + \ldots + (-1)^{n+1}\mathbb{P}(\cap_{1\leq i\leq n} A_i) + \mathbb{P}(A_{n+1}) -$

$\left(\sum_{i=1}^{n}\mathbb{P}(A_i \cap A_{n+1}) - \sum_{1\leq i<j\leq n}\mathbb{P}(A_i \cap A_j \cap A_{n+1}) + \ldots + (-1)^{n+1}\mathbb{P}(\cap_{1\leq i\leq n+1} A_i)\right) =$

$\sum_{i=1}^{n+1}\mathbb{P}(A_i) - \sum_{1\leq i<j\leq n+1}\mathbb{P}(A_i \cap A_j) + \ldots + (-1)^{n+2}\mathbb{P}(\cap_{1\leq i\leq n+1} A_i)$