Probability (Sheet #6)

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1. (a) $G_X(s) = \mathbb{E}[s^X] = \sum_{i=0}^{\infty} s^i \mathbb{P}(X = i) = s^k$

   (b) $G_{mY+n}(s) = \mathbb{E}[s^{mY+n}] = \mathbb{E}[(s^m)^Y s^n] = G_Y(s^m)s^n$.

2. (a) The probability of having m successes and $k-m$ failures is $p^m(1-p)^{k-m}$. As the last trial was a success, we have $m-1$ successes in the first $k-1$ trials, and there are $\binom{k-1}{m-1}$ ways of choosing them. Hence the probability is $\binom{k-1}{m-1}p^m(1-p)^{k-m}$.

   (b) $Y = X_1 + X_2 + \ldots + X_m$ where $X = X_i$ is the number of trials until getting a success.

   Therefore $G_Y(s) = G_X(s)^m = \left(\frac{ps}{1-(1-p)s}\right)^m$

3. $G_Z(s) = G_N(G_X(s))$.

   $\mathbb{E}[Z] = G'_Z(1) = G'_N(G_X(1))G'_X(1) = G'_N(1)\mathbb{E}(X_1) = \mathbb{E}(X_1)\mathbb{E}(N)$.

   $\operatorname{var}(Z) = G''_Z(1) + G'_Z(1) - (G'_Z(1))^2 = (G'_N(1)G'_X(1))' + G'_N(1)G'_X(1) - (G'_N(1)G'_X(1))^2 = G''_N(1)G'_X(1) + G'_N(1)G''_X(1) + G'_N(1)G'_X(1) - (G'_N(1)G'_X(1))^2 = (G'_X(1))^2(G''_N(1) + G'_N(1) - (G'_N(1^2))) + (G'_N(1))(G''_X(1) + G'_X(1) - (G'_X(1))^2) = (\mathbb{E}(X_1))^2 \operatorname{var}(N) + \mathbb{E}[N] \operatorname{var}(X_1)$

   $\operatorname{var}(Z) = \operatorname{var}(N)(\mathbb{E}[X_1])^2 + \mathbb{E}[N] \operatorname{var}(X_1) = \lambda p^2 + \lambda p(1-p) = \lambda p$

4. $G_N(s) = \sum_{k=0}^{n} s^k(1-p)p^n = \frac{1-p}{1-ps}$

   (a) $G_R(s) = G_N(\frac{1}{2} + \frac{1}{2}s) = \frac{1-p}{1-p(\frac{1}{2}+\frac{1}{2}s)} = \frac{2-2p}{2-p-ps} = 2(1-p)(2-p-ps)^{-1} = 2(1-p)(2-p)^{-1}(1-\frac{ps}{2-p})^{-1} = 2\frac{1-p}{2-p}\sum_{k=0}^{\infty}(\frac{ps}{2-p})^k$

   $\mathbb{P}(R = r) = 2\frac{1-p}{2-p}(\frac{p}{2-p})^r$

   (b) $\mathbb{P}(R = r | N = n) = \binom{n}{r}\frac{1}{2}^n$

   $\mathbb{P}(N = n | R = r) = \frac{\mathbb{P}(N = n \cap R = r)}{\mathbb{P}(R = r)} = \frac{\mathbb{P}(R = r | N = n)\mathbb{P}(N = n)}{\mathbb{P}(R = r)} = \frac{\binom{n}{r}\frac{1}{2}^n(1-p)p^n}{2\frac{1-p}{2-p}(\frac{p}{2-p})^r} = \binom{n}{r}\frac{1}{2^{n+1}}p^{n-r}(2-p)^{r+1}$

5. $G(s) = \frac{1}{12} + \frac{2}{3}s + \frac{1}{4}s^2$

   After two minutes, $G_2(s) = G(G(s))$.

   $\mathbb{P}(X_2 = 0) = G_2(0) = G(\frac{1}{12}) = \frac{1}{12} + \frac{2}{36} + \frac{1}{576} = \frac{48+32+1}{576} = \frac{9}{64}$

6. (a) Similarly to 4a we get $G_{X_n}(s) = \frac{p}{1-(1-p)s}$.

   $G'_{X_n}(s) = p\frac{1-p}{(1-(1-p)s)^2}$.

   $\mu = \mathbb{E}(X_n) = G'_{X_n}(1) = p\frac{1-p}{p^2} = \frac{1-p}{p} = \frac{q}{p}$.

(b) Base case: $G_1(s) = \frac{p}{1-(1-p)s} = \frac{p}{q+p-qs} = p\frac{q-p}{(q^2-p^2)-qs(q-p)}$.

Suppose the statement is true for n.

$G_{n+1}(s) = G(G_n(s)) = G\left(p\dfrac{(q^n-p^n)-qs(q^{n-1}-p^{n-1})}{(q^{n+1}-p^{n+1})-qs(q^n-p^n)}\right) =$

$\dfrac{p}{1-pq\frac{(q^n-p^n)-qs(q^{n-1}-p^{n-1})}{(q^{n+1}-p^{n+1})-qs(q^n-p^n)}} =$

$p\dfrac{(q^{n+1}-p^{n+1})-qs(q^n-p^n)}{(q^{n+1}-p^{n+1})-qs(q^n-p^n)-pq((q^n-p^n)-qs(q^{n-1}-p^{n-1}))} =$

$p\dfrac{(q^{n+1}-p^{n+1})-qs(q^n-p^n)}{(q^{n+1}-p^{n+1})-qs(q^n-p^n)-(pq^{n+1}-p^{n+1}q)+qs(pq^n-p^nq)} =$

$p\dfrac{(q^{n+1}-p^{n+1})-qs(q^n-p^n)}{(q^{n+1}-p^{n+1})-qs(q^n-p^n)-(q^{n+1}-p^{n+1})+(q^{n+2}-p^{n+2})+qs(q^n-p^n)-qs(q^{n+1}-p^{n+1})} =$

$p\dfrac{(q^{n+1}-p^{n+1})-qs(q^n-p^n)}{(q^{n+2}-p^{n+2})-qs(q^{n+1}-p^{n+1})}$

(c) $\mathbb{P}(X_n = 0) = G_n(0) = p\frac{q^n-p^n}{q^{n+1}-p^{n+1}} = p\frac{\mu^n-1}{q\mu^n-p} = \frac{\mu^n-1}{\mu^{n+1}-1} = \frac{1-\frac{1}{\mu^n}}{\mu-\frac{1}{\mu^n}}$

If $\mu < 1$, $\mu^n \to 0$ so $\alpha = \lim_{n\to\infty}\frac{\mu^n-1}{\mu^{n+1}-1} = \frac{-1}{-1} = 1$ (which means the population will always die out).

If $\mu > 1$, $\alpha = \lim_{n\to\infty}\frac{1-\frac{1}{\mu^n}}{\mu-\frac{1}{\mu^n}} = \frac{1}{\mu} \neq 1$ (which means the population will not always die out).

(d) For $\alpha = 1$ we get $1 = G(1)$, trivially true.

For $\alpha = \frac{1}{\mu} = \frac{p}{q}$, $G(\frac{p}{q}) = \frac{p}{1-q\frac{p}{q}} = \frac{p}{1-p} = \frac{p}{q}$.