Probability (Sheet #7)

Marius Gavrilescu

1. (a) $F_X(x) = \int_0^x f_1(u)du = \begin{cases} \frac{c}{2}x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x \le 0 \\ \frac{c}{2} & \text{if } x \ge 1 \end{cases}$

   We have $\frac{c}{2} = 1$ so $c = 2$.

   (b) $\int_{-\infty}^{\infty} e^{-x-e^{-x}}dx = e^{-e^{-x}}\Big|_{-\infty}^{\infty} = 1 - 0 = 1$ and $e^{-x-e^{-x}} > 0 \forall x \in \mathbb{R}$, so f_2 is a probability density function.

   $F_2(x) = \int_{-\infty}^{x} f_2(x) = e^{-e^{-x}}\Big|_{-\infty}^{x} = e^{-e^{-x}}$.

2. (a) $F_U(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

   (b) $\mathbb{E}[U] = \int_{-\infty}^{\infty} x f_U(x)dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

   $\text{var}(U) = \left(\int_{-\infty}^{\infty} x^2 f_U(x)dx\right) - (\mathbb{E}[U])^2 = \frac{x^3}{3}\Big|_0^1 - \frac{1}{4} = \frac{1}{12}$

   (c) $\mathbb{P}(U < a|U < b) = \frac{\mathbb{P}(U<a \cap U<b)}{\mathbb{P}(U<b)} = \frac{\mathbb{P}(U<a)}{\mathbb{P}(U<b)} = \frac{F_U(a)}{F_U(b)} = \frac{a}{b}$

3. (a) $\mathbb{P}(X \le x) = F_X(x) = \int_{-\infty}^{x} f_X(u)du = -e^{-\lambda u}\Big|_0^x = 1 - e^{-\lambda x}$

   (b) $\mathbb{P}(a \le X \le b) = F_X(b) - F_X(a) = e^{-\lambda a} - e^{-\lambda b}$

   (c) $\sin X > \frac{1}{2} \iff 2\pi k + \frac{\pi}{6} < X < 2\pi k + 5\frac{\pi}{6}$.

   $$\mathbb{P}(sinX > \frac{1}{2}) = \sum_{k=0}^{\infty}(F_X(2\pi k + 5\frac{\pi}{6}) - F_X(2\pi k + \frac{\pi}{6})) = \sum_{k=0}^{\infty}(1 - e^{-\lambda(2\pi k + 5\frac{\pi}{6})} - 1 + e^{-\lambda(2\pi k + \frac{\pi}{6})}) =$$
   $$e^{-\lambda\frac{\pi}{6}}(\sum_{k=0}^{\infty} e^{-2\lambda\pi k}) - e^{-5\lambda\frac{\pi}{6}}(\sum_{k=0}^{\infty} e^{-2\lambda\pi k}) = (e^{-\lambda\frac{\pi}{6}} - e^{-5\lambda\frac{\pi}{6}})\frac{1}{1 - e^{-2\lambda\pi}} = e^{-\lambda\frac{\pi}{6}}\frac{1 - e^{-2\lambda\frac{\pi}{3}}}{1 - e^{-2\lambda\pi}}.$$

   (d) $\mathbb{P}(X > x) = e^{-\lambda x}$

   $\mathbb{P}(X > x + a|X > a) = \frac{\mathbb{P}(X>x+a \cap X>a)}{\mathbb{P}(X>a)} = \frac{e^{-\lambda(x+a)}}{e^{-\lambda a}} = e^{-\lambda x} = \mathbb{P}(X > x)$.

4. Let $Z = \frac{X-315}{131}$. Then $X = 131Z + 315$.

   (a) $\mathbb{P}(X \le 300) = \mathbb{P}(131Z + 315 \le 300) = \mathbb{P}(Z \le -0.115) = \Phi(-0.115) = 0.454$

   (b) $\mathbb{P}(300 \le X \le 500) = \mathbb{P}(X \le 500) - \mathbb{P}(X \le 300) = \mathbb{P}(131Z + 315 \le 500) - 0.454 = \mathbb{P}(Z \le 1.412) - 0.454 = 0.921 - 0.454 = 0.467$

   (c) Let p_i be the probability that the ith smoker has a nicotine level higher than 500 and all the other smokers have a nicotine level lower than 500.

   Then $p_i = 0.079 \cdot (0.921)^{19} = 0.01654$. The probability of at most one smoker having a nicotine level higher than 500 is $0.0921^{20} + 20 \cdot 0.01654 = 0.524$.

5. (a) $F_X(n) = \sum_{k=1}^{n}(1-p)^{k-1}p = \sum_{k=0}^{n-1}(1-p)^k p = 1-(1-p)^n$.
$F_X(k) = 1-(1-p)^{\lfloor k \rfloor}$

(b) $F_X(nx) = 1-(1-\frac{\lambda}{n})^{nx} = 1-(1-\frac{\lambda}{n})^{-\frac{n}{\lambda}-\lambda x}$
$\lim\limits_{n->\infty} F_X(nx) = 1-e^{-\lambda x}$
$X/n \sim \text{Exp}(\lambda)$ as $n \to \infty$

(c) $F_Y(x) = 1-e^{-\lambda x}$
$p_{\lceil Y \rceil}(x) = F_Y(x) - F_Y(x-1) = e^{-\lambda(x-1)} - e^{-\lambda x} = e^{-\lambda(x-1)}(1-e^{-\lambda})$
Therefore $\lceil Y \rceil \sim \text{Geom}(1-e^{-\lambda})$.

6. $\mathbb{E}[X] = \int_0^1 xf(x)dx = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^\alpha (1-x)^{\beta-1}dx = \int_0^1 \frac{\alpha}{\alpha+\beta}\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)} x^\alpha (1-x)^{\beta-1}dx = \frac{\alpha}{\alpha+\beta}$.
$\text{var}(X) = (\int_0^1 x^2 f(x)dx) - (\mathbb{E}[X])^2 = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} - \frac{\alpha^2}{(\alpha+\beta)^2} = \frac{\alpha(\alpha+1)(\alpha+\beta)-\alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.