Set Theory (Sheet #2)

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1. (a) We can comprehend it. Note that $\bigcup\bigcup R$ exists by unions and is the set of all sets that appear on either the left or the right side of the relation.

   $dom(R) = \{x \in \bigcup\bigcup R : \exists y \in \bigcup\bigcup R.\langle x, y\rangle \in R\}$

   (b) A relation $f \subseteq X \times Y$ is a function if $\phi(f) = \forall x.x \in X \to \exists y.y \in Y \wedge \langle x, y\rangle \in f \wedge \forall z.\langle x, z\rangle \in f \to z = y$ meaning "each element of X maps to a unique element of Y".

   A function is surjective if $\psi(f) = \forall y.y \in Y \to \exists x.x \in X \wedge \langle x, y\rangle$ meaning "each element of Y maps to (at least) one element of X".

   We can therefore comprehend this set as: $\{f \in \mathcal{P}(X \times Y) : \phi(f) \wedge \psi(f)\}$.

   (c) A strict total order is a relation $R \subseteq X \times X$ satisfying these properties:

   $\phi_1(R) = \forall x \in X.\langle x, x\rangle \notin R$
   $\phi_2(R) = \forall x, y, z \in X.(\langle x, y\rangle \in R \wedge \langle y, z\rangle \in R) \to \langle x, z\rangle \in R$
   $\phi_3(R) = \forall x, y \in X.\langle x, y\rangle \in R \leftrightarrow \langle y, x\rangle \notin R$.

   We can therefore comprehend this set as: $\{R \in \mathcal{P}(X \times Y) : \phi_1(R) \wedge \phi_2(R) \wedge \phi_3(R)\}$.

2. We will prove $\phi(n) = n \notin n$ by induction on $\mathbb{N}$.

   Suppose $n = 0$. Then $\phi(n) = \emptyset \notin \emptyset$ which is true since the empty set is empty.

   Now suppose $\phi(n)$. Then $n^+ = n \cup \{n\}$. Assume $n^+ \in n^+$. Then $n^+ \in n$ or $n^+ \in \{n\}$. In the latter case $n^+ = n$, which means $n \cup \{n\} = n$, so $n \in n$. This contradicts the IH, so we must be in the former case ($n^+ \in n$). As n is transitive we know $n^+ \subseteq n$ and since $n \in n^+$ we get $n \in n$, which also contradicts the IH. So our assumption must be wrong, which means $\phi(n^+)$.

   By induction $n \notin n$ for all $n \in \omega$.

3. We use recursion where $X = \omega \times \omega, x_0 = \langle 0, 1\rangle$ and $g(\langle a, b\rangle) = \langle a^+, a^+ \cdot b\rangle$. Then we know the function $f : \omega \to X$ with $f(0) = x_0$ and $f(n^+) = g(f(n))$ exists.

   This function satisfies $f(n) = \langle n, n!\rangle$. From the lecture notes we know its range is a set, and we can observe the range is in fact the function from n to $n!$.

4. Base case: $0 \cdot 0 = 0$, true. Induction step: assume $0 \cdot n = 0$. Then $0 \cdot n^+ = 0 \cdot n + 0 = 0 + 0 = 0$.

   Base case: $m^+ \cdot 0 = m \cdot 0 + 0$ is $0 = 0 + 0$, true. Induction step: assume $m^+ \cdot n = m \cdot n + n$. Then $m^+ \cdot n^+ = m^+ \cdot n + m^+ = m^+ \cdot n + m + 1 = m \cdot n + n + m + 1 = m \cdot n^+ + n + 1 = m \cdot n^+ + n^+$.

   Base case: $m \cdot 0 = 0 \cdot m$, true (both sides are 0). Induction step: assume $m \cdot n = n \cdot m$. Then $m \cdot n^+ = m \cdot n + n = n \cdot m + n = n^+ \cdot m$.

5. Base case: $0 = 2 \cdot 0$ so 0 is even (so 0 is even or odd). Induction step: assume n is even or odd. If n is even, then $n = 2 \cdot k$ for some k and so $n^+ = n + 1 = 2 \cdot k + 1$ so n^+ is odd. If n is odd, then $n = 2 \cdot k + 1$ for some k and so $n^+ = n + 1 = 2 \cdot k + 1 + 1 = 2 \cdot k + 2 = 2 \cdot (k + 1)$. So n^+ is even. Therefore n^+ is also even or odd. By induction all n are even or odd.

   Assume there exist numbers that are both even and odd. Let n be the smallest such number. Then either $n = 0$ or $n = m^+$ for some m. But 0 cannot be odd as $2 \cdot k + 1$ is at least 1, so $n \neq 0$. Therefore $n = m^+$ for some m.

   Observe that since n is both odd and even we have some k and t such that $n = 2 \cdot k$ and $n = 2 \cdot t + 1$. Observe $k \neq 0$, so $k = v^+$. Then $m^+ = n = 2 \cdot v^+ = 2 \cdot v + 2 = 2 \cdot v + 1 + 1 = (2 \cdot v + 1)^+$ and $m^+ = 2 \cdot t + 1 = (2 \cdot t)^+$ and so $m = 2 \cdot v + 1$ and $m = 2 \cdot t$. So m is both odd and even while it is smaller than n, which contradicts the assumption that n was the smallest such number.

6. (a) Clearly ω is a set and $0 \in \omega$. The successor function is one-to-one (if $x^+ = y^+$ that means $x \cup \{x\} = y \cup \{y\}$ and so $x \subseteq y$ and $y \subseteq x$ so $x = y$), and its range does not include 0 ($x \in x^+$ but 0 is the empty set).

   The principle of induction in this triple is exactly induction on ω, which has been proved in the lecture notes.

   Hence $(\omega, x \to x^+, 0)$ is a Peano system.

   (b) We know A is a set, $a_0 \in A$, and $s : A \to A$. By recursion there exists a unique function $f : \omega \to A$ with $f(0) = x_0$ and $f(n^+) = s(f(n))$ for all $n \in \omega$. f already satisfies all properties other than being a bijection.

   Let $S = Im(f)$. Observe $S \subseteq A$, $f(0) = a_0 \in S$, and for all $a \in S$ we know there must be a b such that $f(b) = a$ and then $f(b^+) = s(a)$, so $s(a) \in S$. Thus by the principle of induction $S = A$, so f is surjective.

   Now suppose f is not injective. Let $a < b$ be the smallest natural numbers such that $f(a) = f(b)$. Then observe that $f(b^+) = s(f(b)) = s(f(a)) = f(a^+)$ and by induction $f(b + k) = f(a + k)$. So f is stuck in a loop. Therefore $Im(f)$ doest not contain elements other than $\{f(0), f(1), \ldots, f(b)\}$, which means $Im(f)$ is a finite set. But we know $Im(f) = A$ from the previous paragraph, which means A is finite. But then the function $s : A \to A$ has to be one-to-one and not include a_0 in its range, which is impossible ($|A|$ inputs map to $|A - 1|$ possible outputs). Therefore f must be injective.

   As f is both injective and surjective, it is bijective. Thus it satisfies all the needed properties.

7. (a) Recursion on ω, class form: let $x_0 = X_0$ and the class function $y = \bigcup x$. There is a set Y and function $f : \omega \to Y$ with $f(0) = x_0$ and $\phi(f(n), f(n^+)) \forall n \in \omega$.

   By induction we can show $f(i) = X_i$, as $f(0) = X_0$ and if $f(n) = X_n$ then $f(n^+) = \bigcup X_n = X_{n^+}$.

   As Y is a set, $T(X) = \bigcup Y$ exists by unions.

   (b) Suppose $x \in T(X)$. Then there must exist an i such that $x \in X_i$. But $X_{i+1} = \bigcup X_i$ so $x \subseteq X_{i+1}$ and $X_{i+1} \subseteq T(X)$ so $x \subseteq T(X)$. Therefore $T(X)$ is transitive.

   (c) From the definition of the union, $T(X) = \bigcup\{X_0, X_1, \ldots\}$ must contain every set that is an element of at least one X_i. Therefore $T(X)$ must contain every element of $X_0 = X$. Thus $X \subseteq T(X)$.

   (d) We use induction to prove $\phi(n) = X_n \subseteq Y$.

   Base case: $X_0 = X \subseteq Y$.

   Induction step: Assume $X_n \subseteq Y$. This means that for any $x \in X_n$ we have $x \in Y$. But Y is transitive, so $x \subseteq Y$. Since every element of X_n is a subset of Y, $\bigcup X_n \subseteq Y$. But then $X_{n^+} = \bigcup X_n \subseteq Y$.

   By induction we know all $X_n \subseteq Y$. But then $\bigcup\{X_1, X_2, \ldots\} \subseteq Y$ so $T(X) \subseteq Y$.

   (e) $X \subseteq X$ and X is transitive, so by part (iv) $T(X) \subseteq X$. By part (iii) $X \subseteq T(X)$, so $T(X) = X$.

8. (a) Each of these sets is finite and transitive, so their transitive closures are equal to the sets themselves (and therefore are finite).

   (b) Observe that $Y \subseteq T(Y)$ (by 7.(iii)) and $T(Y)$ is transitive (by 7.(ii)) so if $X \subseteq Y$ then $X \subseteq T(Y)$ and so by 7.(iv) we have $T(X) \subseteq T(Y)$.

   Therefore if Y is a hereditarily finite set and $X \subseteq Y$ then $T(X) \subseteq T(Y)$. Since $T(Y)$ is finite, $T(X)$ is also finite and so X must be hereditarily finite.

   Now if $x \in Y$ then $x \subseteq \bigcup Y$ so $x \subseteq T(Y)$. Since $T(Y)$ is transitive we know $T(T(Y)) = T(Y)$ which is finite, so $T(Y)$ is hereditarily finite. Therefore x is a subset of a hereditarily finite set, so x is also hereditarily finite.

   (c) The empty set, $\emptyset$, is transitive and finite therefore it is part of the class, so the class contains an empty set. ZF2 holds in H.

   Hereditarily finite sets are sets, so they are equal if and only if they have the same elements. ZF1 holds in H.

   Observe that $T(a \cup b) = (a \cup b) \cup \bigcup(a \cup b) \cup \ldots = a \cup \bigcup a \cup \ldots \cup b \cup \bigcup b \cup \ldots = T(a) \cup T(b)$.

   If a, b are hereditarily finite sets, then $c = \{a, b\}$ is a set by pairs, and $T(c) = c \cup (a \cup b) \cup T(a \cup b) = c \cup a \cup b \cup T(a) \cup T(b)$. All these sets are finite, so their union is also finite. Therefore c is a hereditarily finite set.

   If a is a hereditarily finite set then $T(\bigcup a) = \bigcup\limits_{b \in a} T(b)$ which is a union of finite sets, which is finite. So a is hereditarily finite.

   If a is a hereditarily finite set then any set b that can be comprehended from a using the comprehension scheme must be a subset of b, so it must be hereditarily finite.

   (d) No. ω is a transitive set, so $T(\omega) = \omega$, which is not finite.

   (e)

   (f) ZF1 and ZF2 hold as above (all elements of K are still sets).

   If a, b are hereditarily countable sets, then $c = \{a, b\}$ is a set by pairs and $T(c) = c \cup (a \cup b) \cup T(a \cup b) = c \cup a \cup b \cup T(a) \cup T(b)$.

   These sets are all countable so their union is countable, so c is also hereditarily countable. Therefore pairs holds.

   Unions holds because $T(\bigcup A) = \bigcup T(A)$, all of $T(A)$ are countable sets, and $A \subseteq T(A)$ so it is countable, and a countable union of countable sets is countable.

   If a is hereditarily countable then any set b comprehended from a satisfies $b \subseteq a$ and therefore $T(b) \subseteq T(a)$, so $T(b)$ is countable meaning b is hereditarily countable. Thus the comprehension scheme holds.

   Infinity holds because $\omega \in K$ and ω is inductive.

   Powerset does not hold as $\omega \in K$ but $\mathcal{P}(\omega)$ is uncountable and therefore $T(\mathcal{P}(\omega)$ is also uncountable, so $\mathcal{P}(\omega)$ is not hereditarily countable.

   (g)